Technische Universität München
Zentrum Mathematik

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Sheet 2

Algebraic Geometry

To be handed in October 31, before the seminar talk.

We fix a topological space X. By a sheaf (resp. a morphism of sheaves), we always means a sheaf of *abelian groups* (resp. a morphism of sheaves of abelian groups).

Exercise 1. Let $\mathcal{F}$ be a sheaf on X. Let $U \subset X$ be an open subset, and let $s, t \in \mathcal{F}(U)$ be two sections. For all $x \in U$, we note $s_x, t_x \in \mathcal{F}_x$ the germ of s, t, respectively. Show that the set $\{x \in U | s_x = t_x\}$ is open.

Exercise 2. Let k be an algebraic closed field. Let $I \subset k[x_1, ..., x_n]$ be an ideal, and $Y \subset \mathbb{A}^n(k)$ the affine variety defined by I. We note $A := k[x_1, ..., x_n]/I$, and $X := \mathrm{Spec} A$. We note X_0 the subset of X of all closed points, equipped with the induced topology.

1. Show that X_0 and Y are canonically homeomorphic.
2. For all topological spaces T, we denote by $\mathrm{Ouv}(T)$ the set of all open subsets of T, and by $\mathrm{Sh}(T)$ the set of all sheaves on T. Show that the map $\mathrm{Ouv}(X) \to \mathrm{Ouv}(X_0), U \mapsto U \cap X_0$ is a bijection. Show that it induces a bijection between $\mathrm{Sh}(X)$ and $\mathrm{Sh}(X_0)$.
3. Let $\mathcal{F}$ be a sheaf on X. Assume that $\mathcal{F}_x = 0$ for all $x \in X_0$. Show that $\mathcal{F} = 0$.

Exercise 3. Let $X \subset \mathbb{C}$ be an open subset.

1. Show that the presheaf $\mathcal{O}_X : U \mapsto \{\text{holomorphic functions on } U\}$ is a sheaf. It is called the *structural sheaf* of X.
2. Show that the differential operator $D : \mathcal{O}_X \to \mathcal{O}_X$, which sends a holomorphic function f to its derivative f', is a surjection of sheaves. Give an example of an open subset $X \subset \mathbb{C}$ such that $D : \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ is not surjective.

Exercise 4. Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. For all open subsets $U \subset X$, let $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ be the morphism defined by φ.

1. Let $\ker \varphi$ be the presheaf defined by $U \mapsto \ker \varphi(U)$ for all U. Show that $\ker \varphi$ is a sheaf. Show that φ is injective, i.e. φ_x is injective for all $x \in X$, if and only if $\ker \varphi = 0$.
2. Let $\mathrm{coker}\, \varphi$ be the sheafification of the presheaf defined by $U \mapsto \mathrm{coker}\, \varphi(U)$. Prove that φ is surjective if and only if $\mathrm{coker}\, \varphi = 0$. Why do we need sheafification here?

In case of questions please send us an email or contact us before or after the seminar/problem session.
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