Technische Universität München
Zentrum Mathematik

WS 2018/19
Sheet 3

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Algebraic Geometry

To be handed in November 7, before the seminar talk.
We fix a topological space X.

Exercise 1. Let $Z \subset X$ be a closed subset of X. We note $U := X - Z$ the complement of Z. Denote $i : Z \to X$, $j : U \to X$ the morphisms of inclusion.

1. Let $\mathcal{H}$ be a sheaf on Z, and $x \in X$. Calculate the stalk $(i_*\mathcal{H})_x$.
2. Suppose $\mathcal{F}$ is a sheaf on U. Define the sheaf $j_!\mathcal{F}$ on X as the sheafification of the following presheaf: $V \mapsto \mathcal{F}(V)$ if $V \subset U$, $V \mapsto 0$ otherwise. Calculate $(j_!\mathcal{F})_x$ for all $x \in X$. Prove the following adjointness
$$\mathrm{Hom}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, j^{-1}\mathcal{G})$$
for any sheaf $\mathcal{G}$ on X.

Exercise 2. Let $(X_i)_{i\in I}$ be an open covering of X. Suppose for each i we have a sheaf $\mathcal{F}_i$ on X_i, and for each i, j in I, we have an isomorphism of sheaves on $X_i \cap X_j$

$$\varphi_{ji} : \mathcal{F}_i|_{X_i \cap X_j} \xrightarrow{\sim} \mathcal{F}_j|_{X_i \cap X_j},$$

which satisfies the following properties:

1. $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$ for each i;
2. $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ on $X_i \cap X_j \cap X_k$ for each i, j, k in I.

Prove that we can glue sheaves: there exists a unique sheaf $\mathcal{F}$ on X, with an isomorphism

$$\theta_i : \mathcal{F}|_{X_i} \xrightarrow{\sim} \mathcal{F}_i$$

for each i, such that $\varphi_{ji} \circ \theta_i = \theta_j$ on $X_i \cap X_j$ for all i and j.

Exercise 3. Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. For every open subset $U \subset X$, we note $\mathcal{F}|_U$ and $\mathcal{G}|_U$ their restrictions on U. Prove that the map $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ defines a sheaf on X. This sheaf is called the *sheaf Hom* and noted as $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ by some authors.

Exercise 4. In this exercise, we assume that $(X, \mathcal{O}_X)$ is a locally ringed space.

1. Suppose $U \subset X$ is an open and closed subset. Show that there exists a unique element $e_U \in \mathcal{O}_X(X)$, such that $e_U|_U = 1$ and $e_U|_{X-U} = 0$.
2. An element a in a ring is called *idempotent* if $a^2 = a$. Show that, the map $U \mapsto e_U$ gives a bijection between open and closed subsets of X and idempotent elements in $\mathcal{O}_X(X)$. (Hint: one may consider the value at each point of X of the idempotent element in $\mathcal{O}_X(X)$.)

In case of questions please send us an email or contact us before or after the seminar/problem session.
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