Technische Universität München
Zentrum Mathematik

WS 2018/19
Sheet 4

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Algebraic Geometry

To be handed in November 14, before the seminar talk.

Exercise 1. Let $(X, \mathcal{O}_X)$ be a scheme and $U \subseteq X$ an open subset. Prove that $(U, \mathcal{O}_X|_U)$ is a scheme.

Exercise 2. Let k, k' be two fields of *different* characteristic. Let $X \neq \emptyset$ be a k-scheme and X' be a k'-scheme. Show that there is no morphism of schemes between X and X'.

Exercise 3. A scheme X is reduced, if for every open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is reduced (i.e. has no nilpotent elements).

1. Show that X is reduced if and only if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.
2. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf on X associated with the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$, the reduced quotient of the ring $\mathcal{O}_X(U)$. Show that $X_{\text{red}} := (X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. It is called the underlying reduced scheme of X.
3. Show that there is a morphism of schemes $i : X_{\text{red}} \to X$ which is a homeomorphism on topological spaces.
4. Let $f : Y \to X$ be a morphism of schemes and let Y be reduced. Show that then there is a morphism of schemes $g : Y \to X_{\text{red}}$ with $f = i \circ g$.

Exercise 4. Let $(X, \mathcal{O}_X)$ be a scheme such that every $x \in X$ has an open neighborhood which meets only finitely many irreducible components of X. Show that the following are equivalent.

1. Every connected component of X is irreducible.
2. X is the disjoint union of its irreducible components.
3. For all $x \in X$ the nilradical of $\mathcal{O}_{X,x}$ is a prime ideal.

In case of questions please send us an email or contact us before or after the seminar/problem session.
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