Technische Universität München WS 2018/19
Zentrum Mathematik Sheet 5

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# Algebraic Geometry

To be handed in November 21, before the seminar talk.

**Exercise 1.**

Let $k$ be a field. Construct a morphism of schemes $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that on $k$-points we have $f((x_1 : x_2)) = (x_1^2 : x_2^2)$.

**Exercise 2.**

1. Let $R$ be a ring. Prove that $\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R) \cong R$.
2. Let $k$ be a field. Let $f$ be a morphism from $\mathbb{P}^n_k$ to an affine scheme $X$. Prove that $f(\mathbb{P}^n_k)$ consists of a single point of $X$.

**Exercise 3.**

Let $R = \bigoplus_{d\geq 0} R_d$ be a graded ring, i.e. each $R_d$ is an abelian group, and for all $d, e \in \mathbb{N}$, the multiplication of $R$ satisfies $R_d R_e \subset R_{d+e}$.

1. Show that an ideal $I \subseteq R$ is generated by homogenous elements if and only if $I = \bigoplus_d (I \cap R_d)$. Such ideals are called homogenous.
2. Show that the sum, product, intersection, and radical of homogeneous ideals are again homogeneous.
3. Let $I$ be a homogenous ideal. Show that $I$ is prime if and only if for all homogeneous elements $f, g \in R$ with $fg \in I$, we have $f \in I$ or $g \in I$.

**Exercise 4.**

In this exercise, we introduce a finiteness property which has not appeared in the seminar (but is of equal importance). We focus on the case of affine schemes. Suppose we have a morphism of rings $\varphi : A \to B$ which induces the morphism $f : \mathrm{Spec}B \to \mathrm{Spec}A$ of affine schemes. We say that $f$ is *finite* if the ring $B$, when regarded as an *A-module* via $\varphi$, is finitely generated. We say that $f$ is *quasi-finite* if for all $\mathfrak{p} \in \mathrm{Spec}A$, the set $f^{-1}(\mathfrak{p})$ is finite. Prove that if $f$ is finite, then it is quasi-finite.

In case of questions please send us an email or contact us before or after the seminar/problem session.
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