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Algebraic Geometry

To be handed in November 28, before the exercise session. Notice that in the next week the seminar talks will be on Tuesday (November 27, 8:30-10 am, 02.04.011) and Friday (02.08.011), and the exercise session will be on Wednesday at 12:15 (02.08.011).

General hint: It is very often helpful to reduce to the affine setting.

Exercise 1.

Prove that every quasi-compact scheme has a closed point.

Exercise 2.

Let k be a field, let $f \in k[x_1, \dots, x_n]$ and let $X = \text{Spec}k[x_1, \dots, x_n]/(f)$. Let k' be a field extension of k and $X' = \text{Spec}k'[x_1, \dots, x_n]/(f)$

1. Translate the property of X to be irreducible/reduced/integral into properties of f .
2. Give an example where X is irreducible, but X' is not.
3. Give an example where X is reduced, but X' is not (Hint: Consider characteristic $p \neq 0$).

Exercise 3.

Let X, Y be integral schemes and $f : X \rightarrow Y$ a morphism such that the generic point of Y lies in the image of f . Show that

1. f induces an inclusion $K(Y) \rightarrow K(X)$.
2. $f^\flat : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an injective morphism of sheaves.

Also prove the converse: If X, Y are integral and $f : X \rightarrow Y$ such that $f^\flat : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is injective, then the generic point of Y lies in the image of f .

Exercise 4.

1. Let $f : Z \rightarrow X$ be a morphism of schemes. Then there is a unique closed subscheme Y of X such that the morphism f factors through the inclusion $i_Y : Y \hookrightarrow X$ and with the following universal property: If Y' is a closed subscheme such that f factors through the inclusion $i_{Y'} : Y' \hookrightarrow X$, then also i_Y factors through $i_{Y'}$. The subscheme Y is called the *scheme-theoretic image* of f .
2. If Z as above is reduced, then Y is the reduced subscheme of X whose underlying topological space is the closure of the (topological) image $f(Z)$.

In case of questions please send us an email or contact us before or after the seminar/problem session.
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