Technische Universität München — WS 2018/19
Zentrum Mathematik — Sheet 9

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Algebraic Geometry

To be handed in December 19, before the lecture.

Exercise 1.

1. Let X be the affine line with double origin (as in Example 3.5.12) and $X \to \mathbb{A}^1_k$ the morphism which is the identity on each of the two copies of $\mathbb{A}^1_k$. Describe $X \times_{\mathbb{A}^1_k} X$.
2. What happens if we replace one of the copies of X by X_1: the affine line with a doubled point at 1?

Exercise 2+3.

1. Let R be a ring, N an R-module and
$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$
an exact sequence of R-modules. Prove that
$$0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N)$$
is exact (where all Hom-sets are in the category of R-modules).
2. Prove that for R-modules M, N, P we have a canonical isomorphism
$$\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, P))$$
3. Use the previous results to show that in the context of 1., the sequence
$$M' \otimes_R N \xrightarrow{f \otimes 1} M \otimes_R N \xrightarrow{g \otimes 1} M'' \otimes_R N \to 0$$
is exact. ("The tensor product with N is a right exact functor.")
4. Find an example for $R = \mathbb{Z}$ showing that in general, injectivity of f does not imply injectivity of $f \otimes 1$. ("The tensor product with N is in general not an exact functor.")

Exercise 4. Suppose k is a field. Let $X = \mathrm{Spec} R$, $Y = \mathrm{Spec} S$ be affine k-schemes and $f : X \to Y$ a morphism of k-schemes.

1. Construct a morphism of k-scheme $\Gamma_f : X \to X \times_k Y$ such that for all field extensions k' of k, the induced map $X_k(k') \to X_k(k') \times Y_k(k')$ is given by $x \mapsto (x, f(x))$. This morphism is called the graph of f.
2. Show that im (Γ_f) is closed.
   Note: This statement is in general false if X, Y are no longer affine! An example could already be constructed from Exercise 1 above.
3. Let $\varphi : S \to R$ be the morphism of rings corresponding to f. Show that the ideal defining im (Γ_f) is generated by the elements of the form $1 \otimes_k a - \varphi(a) \otimes_k 1$ for $a \in S$.

In case of questions please send us an email or contact us before or after the seminar/problem session.
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