AVERAGING FUNCTORS IN FARGUES' PROGRAM FOR GL_n

JOHANNES ANSCHÜTZ

Abstract. These notes accompany my talk at the ZAGA on 29.6.2020. Everything is joint work with Arthur-César Le Bras, and very much in progress.

1. Fargues' conjecture for GL_n

Let p be prime, $n \geq 1$. Set

$$\mathrm{Perf} := \text{category of perfectoid spaces over } \overline{\mathbb{F}}_p.$$

Set

$$\mathrm{Bun}_n$$

as the small v-stack sending $S \in \mathrm{Perf}$ to the groupoid of vector bundles of rank n on the Fargues–Fontaine curve $X_{\mathrm{FF},S}$ relative to S (and the local field $\mathbb{Q}_p$). Set

$$\breve{\mathbb{Q}}_p$$

as the completion of the maximal unramified extension of $\mathbb{Q}_p$.

Then:

- There is a map $\mathrm{GL}_n(\breve{\mathbb{Q}}_p) \to \mathrm{Bun}_n(\overline{\mathbb{F}}_p),\ b \mapsto \mathcal{E}_b$ inducing a bijection
$$B(\mathrm{GL}_n) := \mathrm{GL}_n(\breve{\mathbb{Q}}_p)/\sigma - \text{conjugacy} \cong |\mathrm{Bun}_n|.$$
- For $b \in B(\mathrm{GL}_n)$ write
$$\mathrm{Bun}_n^b$$
for the (locally closed) substack of vector bundles, which are v-locally isomorphic to $\mathcal{E}_b$. Have locally closed inclusion
$$j_b \colon \mathrm{Bun}_n^b \hookrightarrow \mathrm{Bun}_n$$
inducing a stratification
$$\mathrm{Bun}_n = \coprod_{b \in B(\mathrm{GL}_n)} \mathrm{Bun}_n^b.$$
- For $b \in \mathrm{GL}_n(\breve{\mathbb{Q}}_p)$ let G_b be the σ-stabilizer of b (an algebraic group over $\mathbb{Q}_p$).
- $\mathcal{E}_b$ is semistable if and only if b basic. The (open) semistable locus in Bun_n has the description
$$\coprod_{d \in \mathbb{Z}} [*/\underline{G_b(\mathbb{Q}_p)}] \cong \coprod_{b \in B(\mathrm{GL}_n) \text{ basic}} \mathrm{Bun}_n^b = \mathrm{Bun}_n^{\mathrm{sst}} \overset{j}{\hookrightarrow} \mathrm{Bun}_n$$
with $\deg(\mathcal{E}_b) = d$, G_b is an inner form of GL_n and
$$\underline{G_b(\mathbb{Q}_p)} = \mathrm{Aut}(\mathcal{E}_b)$$

Date: 29.06.2020.

the sheaf on Perf associated to the topological group $G_b(\mathbb{Q}_p)$.

For each $\mathcal{E} \in \mathrm{Bun}_m(\overline{\mathbb{F}}_p),\ m \geq 0$, have small v-sheaf

$$\mathrm{BC}(\mathcal{E})\colon \mathrm{Perf} \to (\mathbb{Q}_p - v.s.),\ S \mapsto H^0(X_{\mathrm{FF},S}, \mathcal{E}_S).$$

Set

$$\mathrm{Div}^d := (\mathrm{BC}(\mathcal{O}(d)) \setminus \{0\})/\underline{\mathbb{Q}_p^\times}$$

where $\mathcal{O}(d)$ the line bundle associated to $b = p^{-d} \in \mathrm{GL}_1(\breve{\mathbb{Q}}_p)$. Then:

- Div^d parametrizes "relative Cartier effective divisors of X_{FF} of degree d".
- We have (as v-sheaves)

$$\mathrm{Div}^d = (\mathrm{Div}^1)/S_d$$

- Concretely,

$$\mathrm{Div}^1 = \mathrm{Spd}(\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}}$$

and thus for $\ell \neq p$

$$\begin{array}{ccc} \{\text{finite dimensional continuous } \overline{\mathbb{Q}}_\ell - \text{representations of } W_{\mathbb{Q}_p}\} & \cong & \{\overline{\mathbb{Q}}_\ell - \text{local systems on } \mathrm{Div}^1\} \\ E & \mapsto & \underline{E}. \end{array}$$

Fargues, Scholze: For a small v-stack Y can define a certain full subcategory

$$D_{\mathrm{lis}}(Y, \overline{\mathbb{Q}}_\ell) \subseteq D(Y_v, \overline{\mathbb{Q}}_\ell)$$

and for a morphism $f\colon Y' \to Y$ of small v-stacks (relevant to us) a pair of adjoint functors $(f_\natural, f^*)$

$$f^*\colon D_{\mathrm{lis}}(Y, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(Y', \overline{\mathbb{Q}}_\ell),$$

$$f_\natural\colon D_{\mathrm{lis}}(Y', \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(Y, \overline{\mathbb{Q}}_\ell)$$

with the following key properties:

- Excision holds on Bun_n, i.e., $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ admits a(n infinite) semi-orthogonal decomposition by the categories $D_{\mathrm{lis}}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell), b \in B(\mathrm{GL}_n)$.
- For $b \in B(\mathrm{GL}_n)$ there are equivalences

$$\begin{array}{ccccc} D(\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_b(\mathbb{Q}_p)) & \cong & D_{\mathrm{lis}}([*/\underline{G_b(\mathbb{Q}_p)}], \overline{\mathbb{Q}}_\ell) & \cong & D_{\mathrm{lis}}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell) \\ \pi & & \mapsto & & \mathcal{F}_\pi \end{array}$$

with $\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_b(\mathbb{Q}_p)$ the category of *smooth* $\overline{\mathbb{Q}}_\ell$-representations of $G_b(\mathbb{Q}_p)$.
- f^* is the usual pullback for $f_v\colon Y'_v \to Y_v$.
- If f is ℓ-cohomologically smooth, then $f_\natural = f_!$ (up to shift/twist).
- If $f\colon [*/\underline{G_b(\mathbb{Q}_p)}] \to *$, then $f_\natural$ identifies with group *homology* of smooth $\overline{\mathbb{Q}}_\ell$-representations of $G_b(\mathbb{Q}_p)$.

Conjecture 1.1 (Fargues, only for GL_n and irreducible case)**.** *For each irreducible $W_{\mathbb{Q}_p}$-representation E there exists an object $\mathrm{Aut}_E \in D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ such that*

(1) Aut_E *is a Hecke eigensheaf with eigenvalue* $\underline{E}$.
(2) Aut_E *is cuspidal, in particular* $\mathrm{Aut}_E \cong j_!(j^*\mathrm{Aut}_E) \cong j_*(j^*\mathrm{Aut}_E)$
(3) *For $b \in B(\mathrm{GL}_n)$ basic*

$$j_b^*\mathrm{Aut}_E \in D_{lis}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell) \cong D(\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_b(\mathbb{Q}_p))$$

is the (Jacquet-)Langlands correspondent $\mathrm{LL}_b(E)$ *of* E.

Note: As the (Jacquet-)Langlands correspondence for GL_n is known, (2), (3) force

$$(*)\qquad \mathrm{Aut}_E \cong \bigoplus_{b\in B(\mathrm{GL}_n)\ \mathrm{basic}} j_{b,!}(\mathcal{F}_{\mathrm{LL}_b(E)}).$$

This gives a possible definition of Aut_E, but the Hecke eigensheaf property (1) is far from clear (and probably impossible to check in its full meaning).

For X a smooth, projective, geometrically connected curve over some field, the existence of a Hecke eigensheaf associated an irreducible local system on X is known by Frenkel/Gaitsgory/Vilonen.

Aim of this talk: *Assume the local Langlands correspondence for* GL_n. Can one construct Aut_E by geometric methods? E.g., imitate the constructions of Frenkel/Gaitsgory/Vilonen?

Set $\hat{G} := \mathrm{GL}_{n,\overline{\mathbb{Q}}_\ell}$. By work of progress of Fargues/Scholze on the geometric Satake the category

$$\mathrm{Rep}(\hat{G})$$

of finite dimensional (algebraic) representations of $\hat{G}$ acts on $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$. More precisely, for any finite set I there exists a functor

$$T^I\colon \mathrm{Rep}(G)^{\times I}\times D_{\mathrm{lis}}(\mathrm{Bun}_n,\overline{\mathbb{Q}}_\ell)\to D_{\mathrm{lis}}(\mathrm{Bun}_n\times(\mathrm{Div}^1)^I,\overline{\mathbb{Q}}_\ell),\ (V,\mathcal{F})\mapsto T^I_V(\mathcal{F}),$$

which satisfies (among others) the following condition:

Let $\varphi\colon I\twoheadrightarrow J$ be a surjection. Define

$$\Delta_\varphi\colon \mathrm{Bun}_n\times(\mathrm{Div}^1)^I\to \mathrm{Bun}_n\times(\mathrm{Div}^1)^J$$

as the inclusion of the diagonal, and

$$\mathrm{ten}_\varphi\colon \mathrm{Rep}(\hat{G})^{\times I}\to \mathrm{Rep}(\hat{G})^{\times J},\ (V_i)_{i\in I}\mapsto (\bigotimes_{i\in\varphi^{-1}(j)} V_i)_{j\in J}.$$

Then

$$\Delta_\varphi^*\circ T^I\cong T^J\circ(\mathrm{ten}_\varphi\times \mathrm{Id}_{D_{\mathrm{lis}}(\mathrm{Bun}_n,\overline{\mathbb{Q}}_\ell)}).$$

Moreover, if $I = I_1\coprod I_2$ is a disjoint union, then (with small abuse of notation)

$$T^I_{(V_i)_{i\in I}} = T^{I_1}_{(V_i)_{i\in I_1}}\circ T^{I_2}_{(V_i)_{i\in I_2}}.$$

In particular, if $\varphi\colon\{1,2\}\to\{1\}$, then

$$(T_{V_1}\circ T_{V_2})_{|\mathrm{Bun}_n\times\Delta}\cong T_{V_1\otimes V_2},$$

where $\Delta\subseteq \mathrm{Div}^1\times\mathrm{Div}^1$ is the diagonal.

For an ℓ-adic representation E of $W_{\mathbb{Q}_p}$ and a(n algebraic) representation $\hat{G}(\overline{\mathbb{Q}}_\ell)\to \mathrm{GL}(V)$, set E_V as the composition

$$W_{\mathbb{Q}_p}\to \mathrm{GL}(E)\cong \hat{G}(\overline{\mathbb{Q}}_\ell)\to \mathrm{GL}(V).$$

Now, $\mathcal{F}\in D_{\mathrm{lis}}(\mathrm{Bun}_n,\overline{\mathbb{Q}}_\ell)$ is a Hecke eigensheaf with eigenvalue $\underline{E}$ if for any finite set I and any collection $(V_i)_{i\in I}$ we are given an isomorphism

$$\eta_{(V_i)_{i\in I}}\colon T^I_{(V_i)_{i\in I}}(\mathcal{F})\cong \mathcal{F}\boxtimes(\boxtimes_{i\in I}\underline{E_{V_i}}),$$

which is natural in I, $(V_i)_{i\in I}$. E.g., if $I=\{1\}$ and V_{st} the standard representation, then

$$\eta_{V_{\mathrm{st}}}\colon T_{V_{\mathrm{st}}}(\mathcal{F})\cong\mathcal{F}\boxtimes\underline{E}.$$

For the definition of Aut_E in $(*)$ one might be able to construct $\eta_{V_{\mathrm{st}}}$, but surely not the whole family of natural isomorphisms $\eta_{(V_i)_{i\in I}}$.

Following Beilinson/Drinfeld/Gaitsgory/Arinkin/Hellmann one can hope for a categorical version of Fargues' conjecture. Let

$$X_{\hat{G}}$$

be the Artin stack of n-dimensional ℓ-adic representations of $W_{\mathbb{Q}_p}$, i.e., of homomorphisms $W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$, taken up to conjugacy (cf. Scholze's talk in Moskau/Princeton).

Conjecture 1.2 (Fargues–Scholze)**.** *There exists an equivalence*

$$\mathbb{L}_G\colon D^b(\mathcal{C}oh(X_{\hat{G}})) \xrightarrow{\simeq} D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega.$$

Here $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$ denotes the category of compact objects in $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$. The equivalence $\mathbb{L}_G$ is expected to satisfy the following conditions:

(1) $\mathbb{L}_G$ is equivariant for the action of $\mathrm{Rep}(\hat{G})$ on both sides.
   - Note: When considering $W_{\mathbb{Q}_p}$ we implicitly chose a (completed) algebraic closure C of $\mathbb{Q}_p$, i.e., a point of Div^1. Using the crucial invariance
   $$D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \cong D_{\mathrm{lis}}(\mathrm{Bun}_n \times \mathrm{Spd}(C^\flat), \overline{\mathbb{Q}}_\ell),$$
   we obtain an action $T\colon \mathrm{Rep}(\hat{G}) \times D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$.
   - For the LHS note that we have a morphism
   $$f\colon X_{\hat{G}} \to [\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/\hat{G}]$$
   and an induced monoidal functor $f^*\colon \mathrm{Rep}(\hat{G}) \to \mathrm{Perf}(X_{\hat{G}})$. Thus, $V \in \mathrm{Rep}(\hat{G})$ acts on the LHS simply by tensoring with the vector bundle $f^*(V)$.

(2) For E an irreducible ℓ-adic representation of $W_{\mathbb{Q}_p}$
$$\mathbb{L}_G(k(E)) \cong \mathrm{Aut}_E$$
with $k(E)$ the *regular* representation of the center $\mathbb{G}_m \cong \hat{Z} \subseteq \hat{G}$ at the closed substack $[\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/\hat{Z}] \subseteq X_{\hat{G}}$ determined by E. The decomposition
$$k(E) \cong \bigoplus_{d\in\mathbb{Z}} k(E)_d,$$
where $k(E)_d$ corresponds to the 1-dimensional representation $x \mapsto x^d$ of $\hat{Z} \cong \mathbb{G}_m$, reflects the decomposition
$$\mathrm{Aut}_E \cong \bigoplus_{b\in B(\mathrm{GL}_n)\ \mathrm{basic}} j_{b,!}(\mathcal{F}_{\mathrm{LL}_b(E)})$$
via $d = \deg(\mathcal{E}_b)$. More precisely,
$$\mathbb{L}_G(k(E)_d) \cong j_{b,!}(\mathcal{F}_{\mathrm{LL}_b(E)}).$$

(3) $\mathbb{L}_G(\mathcal{O}_{X_{\hat{G}}}) \cong \mathcal{W}_\psi$, where $\mathcal{W}_\psi$ is the Whittaker sheaf
$$\mathcal{W}_\psi := j_{1,!}(\mathrm{cInd}_{N(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\psi)$$
for some generic character $\psi\colon N(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^\times$ with $N \subseteq \mathrm{GL}_n$ the standard unipotent subgroup of strictly upper triangular matrices.

In particular, an action,

$$\mathrm{Perf}(X_{\hat{G}}) \times D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell),\ (W, \mathcal{F}) \mapsto W * \mathcal{F},$$

called "the spectral action", of the category

$$\mathrm{Perf}(X_{\hat{G}})$$

of perfect complexes on $X_{\hat{G}}$ on $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ is expected to exist. Of course, the action of $\mathrm{Perf}(X_{\hat{G}})$ should extend the action of $\mathrm{Rep}(\hat{G})$, $\mathbb{L}_G$ should be linear for the actions of $\mathrm{Perf}(X_{\hat{G}})$ on both sides. From

$$\mathbb{L}_G(\mathcal{O}_{X_{\hat{G}}}) \cong \mathcal{W}_\psi$$

one then derives the expectation

$$\mathbb{L}_G(\mathcal{V}) \cong \mathcal{V} * \mathcal{W}_\psi$$

for $\mathcal{V} \in \mathrm{Perf}(X_{\hat{G}})$.

Theorem 1.3 (Fargues–Scholze, in preparation, works for general G)**.** *The spectral action of* $\mathrm{Perf}(X_{\hat{G}})$ *on* $D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ *exists.*

Let E be an *irreducibe* ℓ-adic representation of $W_{\mathbb{Q}_p}$ of rank n. Then $k(E) \in \mathrm{IndPerf}(X_{\hat{G}})$, and we can define

$$\mathrm{Aut}_E := k(E) * \mathcal{W}_\psi$$

as a candidate for Fargues' sheaf associated to E. Then it is formal that Aut_E is a Hecke eigensheaf. In fact, this holds for $k(E) * \mathcal{F}$ for *any* $\mathcal{F} \in D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$. Let $V \in \mathrm{Rep}(\hat{G})$. Then

$$\begin{aligned} T_V(k(E) * \mathcal{F}) &\cong f^*(V) * k(E) * \mathcal{F} \\ &\cong (f^*(V) \otimes_{\mathcal{O}_{X_{\hat{G}}}} k(E)) * \mathcal{F} \\ &\cong (E_V \otimes_{\overline{\mathbb{Q}}_\ell} k(E)) * \mathcal{F} \\ &\cong E_V \otimes_{\overline{\mathbb{Q}}_\ell} k(E) * \mathcal{F} \end{aligned}$$

(compatible with $W_{\mathbb{Q}_p}$-action), and similarly for every finite set I, $V_i \in \mathrm{Rep}(\hat{G})$, $i \in I$.

What can be said about Aut_E, e.g., is $\mathrm{Aut}_E \neq 0$? The problem is that we have a priori no concrete describtion for the spectral action of $k(E) = \bigoplus_{d \in \mathbb{Z}} k(E)_d$.

The key property for the existence of the spectral action is the following. The (full) Hecke action

$$T^I \colon \mathrm{Rep}(\hat{G})^{\times I} \times D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n \times (\mathrm{Div}^1)^I, \overline{\mathbb{Q}}_\ell)$$

induces by pullback along

$$\mathrm{Spa}(C^\flat)^I \to (\mathrm{Div}^1)^I$$

a $W_{\mathbb{Q}_p}$*-equivariant* action of $\mathrm{Rep}(\hat{G})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$, i.e., for any finite set I we have a functor

$$T^{I,\mathrm{equiv}} \colon \mathrm{Rep}(\hat{G})^{\times I} \times D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^{W^I_{\mathbb{Q}_p}}$$

satisfying several compatibilities.

The existence of the spectral action implies in particular, that there exists a lot of endofunctors on $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$.

Where do this come from?

Let I be a finite set, $V_i \in \mathrm{Rep}(\hat{G}), i \in I$, and $E_1, \ldots, E_n$ ℓ-adic representations of $W_{\mathbb{Q}_p}$ (of arbitrary rank). To this data we can construct the endofunctor

$$T_{(V_i)_{i\in I},(E_i)_{i\in I}} \colon D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell),\ \mathcal{F} \mapsto R\Gamma_\natural(W^I_{\mathbb{Q}_p}, \boxtimes_{i\in I}(T^{\mathrm{equiv}}_{V_i}(\mathcal{F}) \otimes E_i)),$$

corresponding to the object

$$R\Gamma_\natural(W^I_{\mathbb{Q}_p}, \boxtimes_{i\in I}(f^* V_i \otimes E_i)) \in \mathrm{Perf}(X_{\hat{G}})$$

(note that the f^*V_i carry a canonical $W_{\mathbb{Q}_p}$-action). Here $R\Gamma_\natural(W^I_{\mathbb{Q}_p}, -)$ denotes (continuous) group homology of $W^I_{\mathbb{Q}_p}$, which by duality agrees with (continuous) group cohomology up to a shift/twist. Instead of

$$k(E) * (-) \colon D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$$

for E an irreducible ℓ-adic representation of $W_{\mathbb{Q}_p}$ of rank n, we can try to understand a closely related, but simpler functor, namely, the "first averaging functor"

$$\mathrm{Av}^1_{E^\vee,n} = T_{V_{\mathrm{st}},E^\vee} \colon D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$$

given by the above construction for $I = \{1\}, V_1 = V_{\mathrm{st}}, E_1 = E^\vee$. Concretely,

$$\mathrm{Av}^1_{E^\vee,n}(\mathcal{F}) = \overrightarrow{h}_\natural(\overleftarrow{h}^*(\mathcal{F}) \otimes \alpha^* E^\vee)$$

for the diagram

$$\begin{array}{ccc} & \mathrm{Mod}^1_n := \{\mathcal{E} \hookrightarrow \mathcal{E}' \text{ fiberwise injective}, \deg(\mathcal{E}') = \deg(\mathcal{E}) + 1\} \xrightarrow{\alpha} \mathrm{Div}^1 & \\ \swarrow^{\overleftarrow{h}} & & \searrow^{\overrightarrow{h}} \\ \mathrm{Bun}_n & & \mathrm{Bun}_n. \end{array}$$

By construction, $\mathrm{Av}^1_{E^\vee,n}$ agrees with the spectral action of the object

$$R\Gamma_\natural(W_{\mathbb{Q}_p}, f^*V_{\mathrm{st}} \otimes E^\vee) \cong k(E(1))_1[1] \oplus k(E)_1 \in \mathrm{Perf}(X_{\hat{G}})$$

(which is a skyscraper sheaf at two distinct points of $X_{\hat{G}}$).

Note that we can define $\mathrm{Av}^1_{E^\vee,n}$ for *any* ℓ-adic representation of $W_{\mathbb{Q}_p}$. It is easy to see that

$$\mathrm{Av}^1_{E^\vee,n} = 0$$

if E is irreducible of rank $> n$.

Remark 1.4. The averaging functors alluded to in the title are

$$\mathrm{Av}^d_{E^\vee,n} := (\mathrm{Av}^1_{E^\vee,n} \circ \ldots \circ \mathrm{Av}^1_{E^\vee,n})^{S_d}$$

for $d \geq 1$, which correspond to the spectral action of the objects

$$(R\Gamma_\natural(W_{\mathbb{Q}_p}, f^*V_{\mathrm{st}} \otimes E^\vee) \otimes^{\mathbb{L}}_{\mathcal{O}_{X_{\hat{G}}}} \ldots \otimes^{\mathbb{L}}_{\mathcal{O}_{X_{\hat{G}}}} R\Gamma_\natural(W_{\mathbb{Q}_p}, f^*V_{\mathrm{st}} \otimes E^\vee))^{S_d} \in \mathrm{Perf}(X_{\hat{G}})$$

for $d \geq 1$. We will actually not use the $\mathrm{Av}^d_{E^\vee,n}$ for $d \geq 2$.

One important property of $\mathrm{Av}^1_{E^\vee,n}$ is its behaviour with respect to constant terms. For each standard parabolic $P \subseteq \mathrm{GL}_n$ with Levi $M \cong \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ there exists the constant term functor

$$\mathrm{CT}_P \colon D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2}, \overline{\mathbb{Q}}_\ell),\ \mathcal{F} \mapsto t_\natural \circ s^*$$

coming from the diagram

$$\mathrm{Bun}_n \xleftarrow{\ s\ } \mathrm{Bun}_P \xrightarrow{\ t\ } \mathrm{Bun}_M \cong \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2}.$$

Lemma 1.5. *For each $\mathcal{F} \in D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ (and arbitrary E) there exists a filtration on $\mathrm{CT}_P \circ \mathrm{Av}^1_{E,n}(\mathcal{F})$ with graded pieces (up to shift/twist) $(\mathrm{Id} \times \mathrm{Av}^1_{E,n_2})(\mathrm{CT}_P(\mathcal{F}))$ and $(\mathrm{Av}^1_{E,n_1} \times \mathrm{Id})(\mathrm{CT}_P(\mathcal{F}))$.*

This implies (by our assumption that E is irreducible), that if P is a proper parabolic, then

$$\mathrm{Av}^1_{E,n} \circ \mathrm{CT}_P = 0.$$

We call an object

$$\mathcal{F} \in D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$$

cuspidal if $\mathrm{CT}_P(\mathcal{F}) = 0$ for all proper parabolics $P \subseteq \mathrm{GL}_n$. It is not difficult to see that each cuspidal object $\mathcal{F}$ is automatically supported on the semistable locus, and associated to complexes of supercuspidal representations of $G_b(\mathbb{Q}_p)$, $b \in B(\mathrm{GL}_n)$ basic, there. With more work, one should be able to prove that it is clean, i.e., $j_! j^* \mathcal{F} \cong j_* j^* \mathcal{F}$.

An argument of Frenkel/Gaitsgory/Vilonen yields that a Hecke eigensheaf for E is automatically cuspidal. This crucially uses that E is irreducible.

Lemma 1.6. *Let $\mathcal{F} \in D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ be a Hecke eigensheaf. Then $\mathcal{F}$ is cuspidal.*

Proof. Let $P \subseteq \mathrm{GL}_n$ be a proper parabolic. By the eigensheaf property of $\mathcal{F}$

$$\mathrm{Av}^1_{E^\vee,n}(\mathcal{F}) \cong R\Gamma_\natural(W_{\mathbb{Q}_p}, E \otimes_{\overline{\mathbb{Q}}_\ell} E^\vee) \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{F}.$$

Now, we can use

$$R\Gamma_\natural(W_{\mathbb{Q}_p}, E \otimes_{\overline{\mathbb{Q}}_\ell} E^\vee) \neq 0$$

and the above observation.

$$\mathrm{CT}_P \circ \mathrm{Av}^1_{E^\vee,n} = 0.$$ □.

In particular,

$$\mathrm{Aut}_E = k(E) * \mathcal{W}_\psi$$

is cuspidal. Using that the local Langlands correspondence is realized in the homology(=$\natural$-pushforward) of the infinite level Lubin-Tate space, it seems to be in reach to prove that the object

$$\mathrm{Aut}_E = k(E) * \mathcal{W}_\psi$$

satisfies all the conditions of Fargues' conjectural sheaf associated with E, i.e., its stalks at semistable points recover the local Langlands correspondence. Namely, we know the Hecke eigensheaf property and thus by 1.6 also cuspidality. In particular,

$$\mathrm{Aut}_E$$

is supported on the semistable locus with fibers given by complexes of supercuspidals there. Consider $b, c \in B(\mathrm{GL}_n)$ basic with $\deg(\mathcal{E}_c) \cong \deg(\mathcal{E}_b) + 1$, and $\pi \in \mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_b(\mathbb{Q}_p)$. Then

$$\mathrm{Av}^1_{E^\vee,n}(j_{b,!}(\mathcal{F}_\pi)) = R\Gamma_\natural(W_{\mathbb{Q}_p}, E^\vee \otimes_{\overline{\mathbb{Q}}_\ell} R\Gamma_\natural(G_b(\mathbb{Q}_p), R\Gamma_\natural(\mathcal{M}_{b,c}, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} \pi)),$$

in
$$D_{\mathrm{lis}}(\mathrm{Bun}_n^c, \overline{\mathbb{Q}}_\ell) \cong D(\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_c(\mathbb{Q}_p)),$$
where
$$\mathcal{M}_{b,c} = \{\mathcal{E}_b \hookrightarrow \mathcal{E}_c\}$$
is the (generalized) infinite level Lubin-Tate space associated with b, c (e.g., if $\mathcal{E}_b \cong \mathcal{O}^n$, $\mathcal{E}_c \cong \mathcal{O}(1/n)$ this is the usual infinite level Lubin-Tate space). In the above formula, we may replace
$$R\Gamma_\natural(\mathcal{M}_{b,c}, \overline{\mathbb{Q}}_\ell)$$
by the supercuspidal part
$$R\Gamma_\natural(\mathcal{M}_{b,c}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sc}}.$$
Let us calculate
$$\mathrm{Av}^1_{E^\vee,n}(\mathcal{W}_\psi).$$
Assume $\mathcal{E}_b) \cong \mathcal{O}^n$, and thus $\mathcal{E}_c \cong \mathcal{O}(1/n)$. Using
$$R\Gamma_\natural(G_b(\mathbb{Q}_p), R\Gamma_\natural(\mathcal{M}_{b,c}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sc}} \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{W}_\psi) \cong R\Gamma_\natural(N(\mathbb{Q}_p), R\Gamma_\natural(\mathcal{M}_{b,c}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sc}}),$$
the knowledge of $R\Gamma_\natural(\mathcal{M}_{b,c}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sc}}$, and the uniqueness of (co-)Whittaker models, and concludes that
$$\mathrm{Av}^1_{E^\vee,n}(\mathcal{W}_\psi) \cong j_{c,!}\mathcal{F}_{\mathrm{LL}_c(E)} \oplus j_{c,!}\mathcal{F}_{\mathrm{LL}_c(E(1))}[1].$$
Considering cohomological degrees and $\mathrm{Av}^1_{E^\vee(-1)}$ one concludes, as desired,
$$k(E)_1 * \mathcal{W}_\psi \cong j_{c,!}\mathcal{F}_{\mathrm{LL}_c(E)}.$$
The irreducibility at all other stalks of Aut_E follows now from the next lemma.

Lemma 1.7. *Let $\mathcal{F} \in D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ be a Hecke eigensheaf with eigenvalue $\underline{E}$. If for some $b \in B(\mathrm{GL}_n)$ basic, the stalk $\mathcal{F}_b := j_b^*\mathcal{F}$ corresponds to an irreducible representation, then this holds for all $c \in B(\mathrm{GL}_n)$ basic.*

Recall that we assumed that E is irreducible.

Proof. Take $b, c \in B(\mathrm{GL}_n)$ basic with $\deg(\mathcal{E}_c) = \deg(\mathcal{E}_b) + 1$. We already know that $\mathcal{F}$ is supported on the semistable locus and that $\mathcal{F}_c$ is a direct sum Let $V := V_{\mathrm{st}}$ be the standard representation of $\hat{G}$. Then
$$T_{V^\vee}(\mathcal{F}_c) \cong \underline{V^\vee} \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{F}_b$$
as $W_{\mathbb{Q}_p}$-equivariant sheaves. The RHS is irreducible (as a $W_{\mathbb{Q}_p}$-equivariant sheaf). Thus, $T_{V^\vee}$ kills all irreducible summands of $\mathcal{F}_c$, except one. But
$$\mathcal{F}_c = T_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}_c) \to T_V(T_{V^\vee}(\mathcal{F}_c))$$
is a split injection, and thus $\mathcal{F}_c$ corresponds to an irreducible representation, placed in some degree. □

Using a geometric version of the Zelevinsky involution one should be to check that $\mathcal{F}_c$ is indeed concentrated in degree 0.

Using arguments of Faltings/Kaletha/Weinstein one should be able to check that the stalks of Aut_E at $c \in B(\mathrm{GL}_n)$ basic with $\deg(\mathcal{E}_c) \neq 1$, are given by $j_{c,!}(\mathcal{F}_{\mathrm{LL}_c(E)}$.

Conjecturally, the category of Hecke eigensheaves in $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ with eigenvalue $\underline{E}$ is equivalent to $D(\overline{\mathbb{Q}}_\ell)$. Thus, conjecturally, Fargues' sheaf is unique up to tensoring with a 1-dimensional $\overline{\mathbb{Q}}_\ell$-vector space.

Let us explain how the above construction of Aut_E relates to the classical construction of Frenkel/Gaitsgory/Vilonen, which rest on the Laumon sheaf $\mathcal{L}_E$ on the stack of torsion sheaves $\mathcal{C}oh_0$ on the projective, smooth, geometrically connected curve X. Set

$$\mathrm{Bun}'_n := \{\Omega_X^{n-1} \hookrightarrow \mathcal{E} \text{ injective with flat cokernel}\}.$$

Using the diagram

$$\begin{array}{ccc} & \mathrm{Mod}'^{,d}_n = \{\Omega_X^{n-1} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{E}',\ \deg(\mathcal{E}') = \deg(\mathcal{E}) + d\} \xrightarrow{\alpha} \mathrm{Coh}_0^d & \\ \swarrow^{\overleftarrow{h}} & & \searrow^{\overrightarrow{h}} \\ \mathrm{Bun}'_n & & \mathrm{Bun}'_n. \end{array}$$

one constructs endofunctors

$$\mathrm{Av}'^{,d}_{E^\vee,n} \colon D(\mathrm{Bun}'_n, \overline{\mathbb{Q}}_\ell) \to D(\mathrm{Bun}'_n, \overline{\mathbb{Q}}_\ell),\ \mathcal{F} \mapsto \overrightarrow{h}_!(\overleftarrow{h}^*(\mathcal{F}) \otimes \alpha^*(\mathcal{L}_{E^\vee})).$$

Starting with a generic character of the standard unipotent one constructs a certain sheaf $\mathcal{K}_\psi \in D(\mathrm{Bun}'_n, \overline{\mathbb{Q}}_\ell)$. Then they consider

$$\mathrm{Aut}'_E := \bigoplus_{d \geq 1} \mathrm{Av}'^{,d}_{E^\vee,n}(\mathcal{K}_\psi)$$

and prove that Aut'_E descents, roughly, to a Hecke eigensheaf on Bun_n. Let $r \colon \mathrm{Bun}'_n \to \mathrm{Bun}_n$ be the natural morphism. If the above constructions (existence of $\mathcal{C}oh_0$, $\mathcal{L}_E$,...) work analogously for the Fargues–Fontaine curve (with Ω_X^1 replaced by $\mathcal{O}$), then one would have

$$r_!(\mathrm{Aut}'_E) = \bigoplus_{d \geq 1} \mathrm{Av}^d_{E^\vee,n}(\mathcal{W}_\psi).$$

By non-obvious additional arguments one should then be able to conclude

$$r^!(\mathrm{Aut}_E) = r^!(k(E) * \mathcal{W}_\psi) \cong \mathrm{Aut}'_E.$$