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Introduction to Algebraic Number Theory
Sheet 7

Exercise 1. Let K be a number field.

(a) Let I be an ideal of $\mathcal{O}_K$ satisfying $I^m = (a)$ for some $m \geq 1$ and $a \in \mathcal{O}_K$. Show that I becomes principal in the field extension $K(\sqrt[m]{a})$ of K in the sense that the ideal $I\mathcal{O}_{K(\sqrt[m]{a})}$ of $\mathcal{O}_{K(\sqrt[m]{a})}$ is principal.

(b) Show that there exists a number field L containing K in which every ideal of $\mathcal{O}_K$ becomes principal.

Exercise 2. The rings $\mathbb{Z}[\sqrt{6}]$ and $\mathbb{Z}[\sqrt{7}]$ are PIDs. Find generators for their ideals $(3, \sqrt{6})$, $(5, 4 + \sqrt{6})$ and $(2, 1 + \sqrt{7})$.
Hint: If an ideal I of $\mathcal{O}_K$ is generated by $a \in \mathcal{O}_K$, then $N(I) = N(a)$.

Exercise 3. Let $d \equiv 1 \pmod 4$ be a square-free integer not equal to 1. Show that the ring $\mathbb{Z}[\sqrt{d}]$ is never a UFD.
Hint: Show that 2 is irreducible but not prime in this ring.