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Introduction to Algebraic Number Theory
Sheet 10

Exercise 1. Let $m \geq 3$ be an integer and ζ a primitive m-th root of unity. Show that for $(k, m) = 1$, the number $\frac{1-\zeta^k}{1-\zeta}$ is a unit in $\mathcal{O}_{\mathbb{Q}(\zeta)}$.

Exercise 2. Let p be an odd prime and ζ a primitive p-root of unity.

(a) Show that $\mathbb{Z}[\zeta]^* = (\zeta)\mathbb{Z}[\zeta + \zeta^{-1}]^*$. (You may use the fact that the polynomial $1 + \ldots + X^{p-1} \in \mathbb{Z}$ is irreducible.)

(b) For $p = 5$, show that $\mathbb{Z}[\zeta]^* = \{\pm\zeta^k(1 + \zeta)^n \mid 0 \leq k < 5,\ n \in \mathbb{Z}\}$.

Hint: For (a), consider the automorphism τ of $\mathbb{Q}(\zeta)$ which sends ζ to ζ^{-1}. Show that $\mathbb{Z}[\zeta + \zeta^{-1}]$ is the set of fixed points of τ in $\mathbb{Z}[\zeta]$. Then show that for every $u \in \mathbb{Z}[\zeta]^*$, the image of $\tau(u)/u$ under every complex embedding of $\mathbb{Q}(\zeta)$ has norm 1.