Prof. Dr. E. Viehmann

Winter 2019/20

Seminar on *Reductive groups over local fields*

There will be a meeting to assign the talks, probably on October 7. If you want to attend the seminar and are not in M11, please send me an email to be put on the mailing list.

The seminar consists of two parts. In the first (talks 1–5) we mainly follow Chapter 2 of [M] for an introduction to Bruhat-Tits buildings. The original articles of Bruhat and Tits are rather long and involved, and assume more knowledge of reductive groups than [M]. An interesting classical, but also advanced summary of the theory can be found in [T].

The second part of the seminar (starting with talk 6) assumes more background (which we review in talks 6 and 7), and also adresses those who are already familiar with Bruhat-Tits theory. We will study an article by Rémy, Thuillier and Werner [RTW] relating the Bruhat-Tits building of a reductive group G over a non-Archimedean field to certain Berkovich spaces associated with the group.

Some of the talks are longer than one session and may be split among two speakers.

Talk 1. BN-pairs.

Define BN-pairs, example: $G = \mathrm{GL}_n(k)$, with B the subgroup of upper triangular matrices and N the monomial matrices (or more generally G the k-values points of a reductive group, B a Borel subgroup and N the normalizer of a maximal torus). ([Bou], IV.2, [M], 2.3)

Define Coxeter systems and the length of an element in a Coxeter group. ([Bou], IV.1)

Prove the Bruhat decomposition ([Bou], IV.2.3, [M], 2.3.1).

Prove that for a BN-pair, the associated pair (W, S) is a Coxeter system ([Bou], IV.2.4).

Prove [M], 2.3.3-2.3.7 and explain them in the example above.

Show that another BN-pair is given by $G = \mathrm{SL}_2(\mathbb{Q}_p)$,

$$B = \{g \in \mathrm{SL}_2(\mathbb{Z}_p) \mid g \bmod p \text{ is upper triangular}\},$$

N the monomial matrices,

$$S = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p \\ -p^{-1} & 0 \end{pmatrix} \right\}.$$

Talk 2. Root systems. Root systems and affine roots, following [M], 2.1, 2.2, also explain all notions for the example of type A_l. Draw a picture for $l = 2$.

Talk 3. The building associated with an affine root system. Define the building associated with a BN pair with affine Weyl group, following [M], 2.4.1-15. Also explain the building associated with the last example of talk 1 ([S], II.1, [G], Ch.19).

Talk 4. The Bruhat-Tits building of a p-adic Lie group. Definition of an affine root structure on a group ([M], 2.5.3). There is a correction (due to Deligne) of axiom V, which should be replaced by

For $\alpha, \beta > 0$, the commutator group $[U_\alpha, U_\beta]$ is contined in the subgroup generated by all U_γ for $\gamma > 0$, and not parallel to α or β.

Definition of groups of p-adic type ([M], beginning of 2.7, including 2.7.1-2.7.3)

Explain in detail the building of $\mathrm{SL}_n(\mathbb{Q}_p)$, and explain all occurring notions for this example ([G], Ch. 18,19, but also part 1 of [BT])

Talk 5. Cartan and Iwasawa decomposition. Prove the Cartan and Iwasawa decompositions, [M], 2.6. Explain 2.6.11 for the example of $\mathrm{SL}_n(\mathbb{Q}_p)$.

Talk 6. Extension of the ground field. Explain the differences between the theory/notation explained in the previous talks and the one used in [RTW], and introduce extended Bruhat-Tits buildings. Introduce functoriality of the Bruhat-Tits building under extension of the ground field, and Bruhat-Tits buildings for not necessarily locally compact ground fields ([RTW], 1.3).

Talk 7. Berkovich spaces. A summary of Berkovich spaces, also explain the notion of Shilov boundary and of the k-analytic space associated with a finitely presented scheme over k°. ([RTW], 1.2)

Talk 8. Realization of buildings. Construct the canonical map $\vartheta : \mathscr{B}^e(G,k) \to G^{\mathrm{an}}$, and discuss its properties. Describe the realization of the building in the flag varieties (of different types) associated with G. ([RTW], 2)

Talk 9/10. Compactifications of buildings. Use the construction explained in the previous talk to define the Berkovich compactification of the Bruhat-Tits building associated with any type t of proper parabolic subgroup of G. Describe the induced group action and the stratification according to t-relevant parabolic subgroups. ([RTW], 3,4)

References

[Bou] N. Bourbaki, Groupes et algèbres de Lie, Chap. 4,5,6.

[BT] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local*, Pub. Math. IHES **41** (1972), 5–251.

[G] P. Garrett, Buildings and classical groups, Chapman and Hall, 1997.

[M] I. G. Macdonald, Spherical functions on a group of p-adic type, Proc. of the Ramanujam Inst., No. 2, University of Madras, 1971.

[S] J.-P. Serre, Arbres, amalgames, SL_2, Astérisque 46, Soc. Math. France, Paris, 1977.

[RTW] B. Rémy, A. Thuillier, A. Werner, *Bruhat-Tits theory from Berkovich's point of view. I – Realizations and compactifications of buildings*, Ann. Sci. Ec. Norm. Sup. **43** (2010) 461–554.

[T] J. Tits, *Reductive groups over local fields*, Proc. Symp. Pure Math. **33** (1979), part 1, 29–69.