Category Theory

What is CT about?

It is about describing certain fundamental patterns which occur in many areas of mathematics:

Eg: I <u>Universal properties</u>:

Eg: Product of two vector spaces:

V_1, V_2 v.s / k field

$\rightsquigarrow V_1 \times V_2$, $\pi_i : V_1 \times V_2 \to V_i$

$(v_1, v_2) \mapsto v_i$

for $i = 1, 2$

$(\ast)$

$\forall$ W v.s $/k$, $f_1 : W \to V_1$, $f_2 : W \to V_2$

k-lin. maps

$\exists!\, f : W \to V_1 \times V_2$ s.t.

$$\begin{tikzcd} & W \arrow[dl, "f_1"'] \arrow[d, "!f"] \arrow[dr, "f_2"] & \\ V_1 & V_1 \times V_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & V_2 \end{tikzcd}$$

commutes, i.e. s.t. $\pi_i \circ f = f_i$ for $i = 1, 2$

This describes $(V_1 \times V_2, \pi_1, \pi_2)$ uniquely up to isom:

Let $(U, \pi_1' : U \to V_1, \pi_2' : U \to V_2)$ be another triple satisfying $(*)$.

Taking $W = U$ in $(*)$ we find a (unique) lin. map $g : U \to V_1 \times V_2$ s.t.

$\pi_i \circ g = \pi_i'$ for $i = 1, 2$.

Reversing the roles of U and $V_1 \times V_2$, we find $h : V_1 \times V_2 \to U$ s.t. $\pi_i' \circ h = \pi_i$.

Then $\pi_i' \circ h \circ g = \pi_i \circ g = \pi_i'$ and so by the unicity of $(*)$ (for $W = U$) we get $h \circ g = id_U$. The other way around we get $g \circ h = id_{V_1 \times V_2}$. So g, h are isomorphisms compatible with the π_i and π_i'.

This describes $V_1 \times V_2$ just by how it maps to other k-v.s.
Same works for products of groups, rings, ...

Many other constructions can be described by univ. properties: direct sums, kernel, cokernel, ...

⇝ One way of looking at categories is that they are the natural setting to talk about universal properties.

II Invariants

General pattern in geometry:
Have some class of complicated objects X e.g. some kind of geometric spaces, and want to understand these by associating to them some simpler invariant $i(X)$.

E.g.:

vector space X $\mapsto$ $\dim X \in \mathbb{N} \cup \{\infty\}$

graph $\mapsto$ # of conn. comp.

top space X $\mapsto$ $\pi_1(X)$ fund. group

$K/\mathbb{Q}$ number field $\mapsto$ $\mathrm{Hom}(K, \mathbb{C})$

finit set of field emb. $K \hookrightarrow \mathbb{C}$

Given such a transformation $X \mapsto i(X)$ often it is extremely useful to also associate to a map $X \to Y$ some kind of map $i(X) \to i(Y)$, this is called functoriality.

Often one also wants to understand how objects constructed by a univ. prop. transform under i.

<u>Def</u>: * A category C consists of

* A set $Ob(C)$ of "objects of C"
* $\forall c, d \in Ob(C)$, a set $Hom(c,d)$ of "maps from c to d", which we call morphism from c to d.

* $\forall c, d, e \in Ob(C)$ a

$$\begin{aligned} Hom(c,d) \times Hom(d,e) &\to Hom(c,e) \\ (g,h) &\mapsto h \circ g \end{aligned}$$

which we think of as composition.

s.t:

* $\exists$ identity morphisms:

  $\forall c \in Ob(C)$ $\exists$ $id_c \in Hom(c,c)$ s.t $\forall d \in Ob(C)$:

  $\forall h \in Hom(c,d)$: $h \circ id_c = h$

  $\forall h \in Hom(d,c)$: $id_c \circ h = h$

* Composition is associative:

  $\forall c, d, e, f \in Ob(C)$

  $\forall g \in Hom(c,d), h \in Hom(d,e), i \in Hom(e,f)$:

  $(i \circ h) \circ g = i \circ (h \circ g)$

Examples:

- Set: $\cdot$ $Ob(Set) = \{ \text{sets} \}$

  $\cdot$ $\forall X, Y$ sets: $Mor(X, Y) = \{ f : X \to Y \}$ with the usual composition

- k a field:
  $Vect_k$ $\cdot$ Objects: v.s. $/k$
  $\cdot$ Morphisms k-linear maps

- Grp: Objects: groups
  Morphisms: group homomorphisms

- Graph: Objects: graphs
  Morphisms: morph. of graphs

All these examples are "concrete": Objects are sets with some add. structure, morphisms are maps respect the add. structure

But there are other categories:

* R ring: $\underline{\mathrm{Mat}}_R$
  * Objects: integers $n \geq 0$
  * $\mathrm{Hom}(n,m) = R^{m \times n}$ matrices

Composition given by matrix mult.:

$$(A \in R^{n \times m}, B \in R^{m \times k}) \mapsto (A \cdot B \in R^{n \times k})$$

Or: *
* Objects: integers $n \geq 0$
* Morph:

$$\mathrm{Hom}(n,m) = \begin{cases} \emptyset & m < n \\ \{*\} & n \leq m \end{cases}$$

with the only possible composition rule

(the same works for any partially ordered set.)