

Category Theory

What is CT about?

It is about describing certain fundamental patterns which occur in many areas of mathematics.

Eg: I Universal properties:

Eg: Product of two vector spaces:

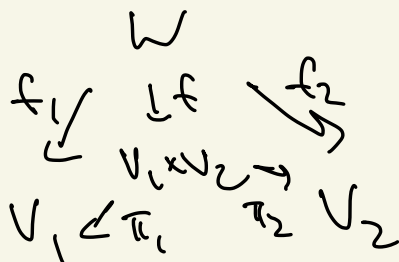
V_1, V_2 v.s / k field

$\leadsto V_1 \times V_2, \pi_i: V_1 \times V_2 \leadsto V_i$
 $(v_1, v_2) \leadsto v_i$

for $i=1,2$

$\forall W$ v.s / $k, f_1: W \rightarrow V_1, f_2: W \rightarrow V_2$
 k -lin. maps

$\exists! f: W \rightarrow V_1 \times V_2$ s.t.



Commutates, i.e. s.t. $\pi_i \circ f = f_i$ for $i=1,2$

This describes $(V_1 \times V_2, \pi_1, \pi_2)$ uniquely up to isom:

Let $(U, \pi'_1: U \rightarrow V_1, \pi'_2: U \rightarrow V_2)$ be another triple satisfying $(*)$.

Taking $W=U$ in $(*)$ we find a (unique) lin. map $g: U \rightarrow V_1 \times V_2$ s.t.
 $\pi_i \circ g = \pi'_i$ for $i=1,2$.

Reversing the roles of U and $V_1 \times V_2$, we find $h: V_1 \times V_2 \rightarrow U$ s.t. $\pi'_i \circ h = \pi_i$.

Then $\pi'_i \circ h \circ g = \pi_i \circ g = \pi'_i$ and so by the unicity of $(*)$ (for $W=U$) we get $h \circ g = \text{id}_U$. The other way around we get $g \circ h = \text{id}_{V_1 \times V_2}$. So g, h are isomorphisms compatible with the π_i and π'_i .

This describes $V_1 \times V_2$ just by how it maps to other U - V 's.

Same works for products of groups, rings, ...

Many other constructions can be described by univ. properties: direct sums, kernel, cokernel, ...

→ One way of looking at categories is that they are the natural setting to talk about universal properties.

II Invariants

General pattern in geometry:

Have some class of complicated objects X
e.g. some kind of geometric spaces,
and want to understand these by
associating to them some simpler
invariant $i(X)$.

Es:

vector space X $\mapsto \dim X \in \mathbb{N} \cup \{\infty\}$
graph $\mapsto \#$ of conn. comp.
top space $X \mapsto \pi_1(X)$ fund group
 K/\mathbb{Q} number field $\mapsto \text{Hom}(K, \mathbb{C})$
finit set of field
emb. $K \hookrightarrow \mathbb{C}$

Given such a transformation $X \mapsto i(X)$
often it is extremely useful to
also associate to a map $X \rightarrow Y$
some kind of map $i(X) \rightarrow i(Y)$,
this is called functoriality.

Often one also wants to understand
how objects constructed by a univ.
prop. transform under i .

Def: * A category C consists of

* A set $Ob(C)$ of "objects of C "

* $\forall c, d \in Ob(C)$, a set $Hom(c, d)$ of "maps from c to d ", which we call morphism from c to d .

* $\forall c, d, e \in Ob(C)$ a

$$Hom(c, d) \times Hom(d, e) \rightarrow Hom(c, e) \\ (g, h) \mapsto h \circ g$$

which we think of as composition.

s.t.:

* \exists identity morphisms:

$$\forall c \in Ob(C) \exists id_c \in Hom(c, c)$$

$$\text{s.t. } \forall d \in Ob(C):$$

$$\forall h \in Hom(c, d): h \circ id_c = h$$

$$\forall h \in Hom(d, c): id_c \circ h = h$$

* Composition is associative:

$$\forall c, d, e, f \in Ob(C)$$

$$\forall g \in Hom(c, d), h \in Hom(d, e), i \in Hom(e, f):$$

$$(i \circ h) \circ g = i \circ (h \circ g)$$

Examples:

* Set : $\text{Ob}(\text{Set}) = \{\text{sets}\}$

• $\forall X, Y \text{ sets: } \text{Mor}(X, Y) = \{f: X \rightarrow Y\}$
with the usual composition

* K a field:

Vect_K

• Objects: V -s. / K

* Morphisms K -linear maps

• Gro :

Objects: groups

Morphisms: group homomorphisms

• Graph :

Objects: graphs

Morphisms: morph. of graphs

All these examples are "concrete":

Objects are sets with some add. structure, morphisms are maps respect the add. structure

But there are other categories:

* Ring: $\underline{\text{Mat}}_{\mathbb{R}}$. Objects: integers $n \geq 0$
Hom(n, m) = $\mathbb{R}^{m \times n}$ matrices

Composition given by matrix mult.:

$$(A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times k}) \mapsto (A \cdot B \in \mathbb{R}^{n \times k})$$

Or: * Objects: integers $n \geq 0$

* Morph:

$$\text{Hom}(n, m) = \begin{cases} \emptyset & m < n \\ \{*\} & n \leq m \end{cases}$$

with the only possible composition rule
(the same works for any partially ordered set.)