Given $G: \mathcal{D} \to \mathcal{C}$, when does it admit a left adjoint?

Say $\mathcal{D}$ loc-small + complete. Then G needs to be continuous. Up to size issues, this is actually sufficient for G to admit a left adjoint:

To construct a l.a. $F: \mathcal{C} \to \mathcal{D}$, for $c \in \mathcal{C}$ we need $Fc \in \mathcal{D}$ with an iso. of functors

$$\mathrm{Hom}(Fc, -) \cong \mathrm{Hom}(c, G-) : \mathcal{D} \to \mathrm{Set}$$

If such a Fc exists for each $c \in \mathcal{C}$, then the Fc are automatically functorial via Yoneda.

To see whether such a Fc exist, we look at the category

$c \downarrow G$: Objects $(d \in \mathcal{D}, c \xrightarrow{f} Gd \in \mathcal{C})$

Morph $(d, f) \to (d', f')$ are morph $d \xrightarrow{h} d'$ s.t.

$$\begin{tikzcd} & Gd \arrow[d, "Gh"] \\ c \arrow[ur, "f"] \arrow[r, "f'"'] & Gd' \end{tikzcd}$$

commutes

If Fc exists, then

$$\mathrm{Hom}(Fc, Fc) \cong \mathrm{Hom}(c, GFc)$$

$$\mathrm{id}_{Fc} \leftrightarrow \eta: c \to GFc$$

gives an object $(Fc, \eta) \in c \downarrow G$

which is initial.

One can check: Fc exists $\Leftrightarrow$ $c \downarrow G$ has an initial obj.

Next there is a natural functor

$$U: c\downarrow G \to D$$
$$(d, f) \mapsto d$$
$$(d,f) \xrightarrow{h} (d',f') \mapsto h$$

$\to$ One can check that $c \downarrow G$ has an initial obj $\Leftrightarrow$ $\lim_{c\downarrow G} U$ exists.
Since D is complete, this is always the case if $c\downarrow G$ is small. In general, to get existence of this limit, we need to be able to replace $c \downarrow G$ by a smaller part over which U has the same limit

Thm (Adjoint Functor Theorem)
$G: D \to C$ as above has a left adjoint if and only if it satisfies the Solution Set Condition: For every $c \in C$ there exists a set of morph.
$\{ f_i : c \to G d_i \mid i \in I \}$ s.t. every $f: c \to Gd$ is of the form $c \xrightarrow{f_i} G d_i \xrightarrow{Gh} Gd$ for some $h: d_i \to d$.

"Pf": • If G has a l.a F then $\{\eta_i : c \to GFc\}$ is such a set.

• If G satisfies the SSC, we can form the above limit over such a set.

$\square$

In most situation, the SSC is satisfied, so we can usually expect continuous functors to have a l.a.

Ex: Existence of tensor products:

$\underset{\mathrm{AbGrp} \to \mathrm{AbGrp}}{A \mapsto A \otimes B}$ is l.a to $C \mapsto \mathrm{Hom}(B, C)$

$\uparrow$ this is cont.

$\leadsto$ for A, B need to look at

$$\{A \to \mathrm{Hom}(B, C) \mid C \in \mathrm{AbGrp}\}$$

For $A \xrightarrow{h} \mathrm{Hom}(B, C)$, let $C' \subseteq C$ be the subgroup gen by $\{h(a)(b) \mid \substack{a \in A \\ b \in B}\}$. So C' is the set of sums $\sum_{i=1}^{n} h(a_i)(b_i)$ for elements $(a_i, b_i) \in A^n \times B^n$. So C' has at most the cardinality of $\prod_{i=1}^{\infty} A^i \times B^i$. Such $C' \in \mathrm{AbGrp}$ form a set, and h factors as $A \to \mathrm{Hom}(B, C') \to \mathrm{Hom}(B, C)$. Hence $\mathrm{Hom}(B, -)$ satisfies the SSC, so $A \mapsto A \otimes B$ exists.

Monoidal categories

We want to axiomatize categories C with a tensor product functor $\otimes : C \times C \to C$ which is associative and has a unit (i.e. an object $1\!\!1 \in C$ s.t. $1\!\!1 \otimes c \cong c$ for all $c \in C$):

<u>Def</u>: A monoidal category consists of:

* a cat. C
* a functor $\otimes : C \times C \to C$
  written as $(c, c') \mapsto c \otimes c'$
* an object $1\!\!1 \in C$
* natural isomorphisms

$$\lambda_c \quad 1\!\!1 \otimes c \xrightarrow{\cong} c$$

$$\rho_c \quad c \otimes 1\!\!1 \xrightarrow{\cong} c$$

$$\alpha_{a,b,c} \quad (a \otimes b) \otimes c \xrightarrow{\cong} a \otimes (b \otimes c)$$

  for all $a, b, c \in C$

which satisfy the following coherence cond.:

* $\forall a, b \in C$ the diagram

$$\begin{tikzcd}
(a \otimes 1\!\!1) \otimes b \arrow[rr, "\alpha_{a,1\!\!1,b}"] \arrow[dr, "\lambda_a \otimes id_b"'] & & a \otimes (1\!\!1 \otimes b) \arrow[dl, "id_a \otimes \lambda_b"] \\
& a \otimes b &
\end{tikzcd}$$

  commutes

$\forall a,b,c,d \in C$ the diagram

$$\begin{tikzcd}[column sep=tiny]
& (a\otimes b)\otimes(c\otimes d) \arrow[dr, "\alpha_{a,b,c\otimes d}"] & \\
((a\otimes b)\otimes c)\otimes d \arrow[ur, "\alpha_{a\otimes b,c,d}"] \arrow[d, "\alpha_{a,b,c}\otimes id_d"'] & & a\otimes(b\otimes(c\otimes d)) \\
(a\otimes(b\otimes c))\otimes d \arrow[rr, "\alpha_{a,b\otimes c,d}"'] & & a\otimes((b\otimes c)\otimes d) \arrow[u, "id_a\otimes\alpha_{b,c,d}"']
\end{tikzcd}$$

commutes.

Ex: * $C = AbGrp$ with usual tensor product, $\mathbb{1} = \mathbb{Z}$ and usual id. + assoc. iso's

* Any cat. C with all finite products can be made into a monoidal cat with $c\otimes d = c\times d$ and $\mathbb{1}$ the final object $\left(= \prod_{\emptyset}\right)$

So eg $(Set, \times)$, $(AbGrp, \times)$

* Any cat. with finite coproducts can be made into a monoidal cat. with $c\otimes d = c \amalg d$ and $\mathbb{1}$ the initial object

Eg $(AbGrp, \oplus)$

Coherence Thm: Let $(\mathcal{C}, \otimes, \ldots)$ be a mon. cat., $c_1, \ldots, c_n \in \mathcal{C}$ and

$P_1 = c_1 \otimes 1\!\!1 \otimes \ldots \otimes c_2 \otimes \ldots 1\!\!1 \quad \otimes c_n$ with some brackets

$P_2 = c_1 \otimes \ldots \otimes 1\!\!1 \otimes c_2 \ldots$ with some brackets

$\leadsto \exists!\ P_1 \xrightarrow{\cong} P_2$ obtained by composing some α?, λ?, ρ?

Def: A symmetric monoidal cat. consists of a mon. cat $(\mathcal{C}, \otimes, \ldots)$ together with nat. iso's

$$\sigma_{a,b} : a \otimes b \xrightarrow{\cong} b \otimes a$$

for all $a, b \in \mathcal{C}$ s.t. the following diagrams commute:

* $$\begin{tikzcd} a \otimes 1\!\!1 \arrow[r, "\sigma_{a,1\!\!1}"] \arrow[dr, "\rho_a"'] & 1\!\!1 \otimes a \arrow[d, "\lambda_a"] \\ & a \end{tikzcd}$$

* $$\begin{tikzcd} (a \otimes b) \otimes c \arrow[r, "\sigma \otimes id_c"] \arrow[d, "\alpha"] & (b \otimes a) \otimes c \arrow[r, "\alpha"] & b \otimes (a \otimes c) \arrow[d, "id \otimes \sigma"] \\ a \otimes (b \otimes c) \arrow[r, "\sigma"] & (b \otimes c) \otimes a \arrow[r, "\alpha"] & b \otimes (c \otimes a) \end{tikzcd}$$

* $$\begin{tikzcd} & b \otimes a \arrow[dr, "\sigma"] & \\ a \otimes b \arrow[ur, "\sigma"] \arrow[rr, "id"'] & & a \otimes b \end{tikzcd}$$

Also $\exists$ notions of (symm.) monoidal functors between (symm.) monoidal cat's. and of monoidal nat. transf. between such functors.

Enriched cat:

Def: $(M, \otimes, \ldots)$ monoidal cat
An M-enriched cat. (or just M-cat) C consists of:

- A class $ob(C)$ of obj.
- $\forall c, d \in C$ an object $Hom(c,d) \in M$
- $\forall a, b, c \in C$ a morph
$$Hom(a,b) \otimes Hom(b,c) \xrightarrow{\circ} Hom(a,c)$$
in M.
- $\forall c \in C$ a morph $1\!\!1 \xrightarrow{id_c} Hom(c,c)$ in M

s.t. $\forall a, b, c, d \in C$ the following diagrams commute:

- $$\begin{tikzcd}
Hom(a,b) \otimes (Hom(b,c) \otimes Hom(c,d)) \arrow[r, "\circ"] & Hom(a,b) \otimes Hom(b,d) \arrow[d, "\circ"] \\
(Hom(a,b) \otimes Hom(b,c)) \otimes Hom(c,d) \arrow[u, "\alpha"] \arrow[r, "\circ"] & Hom(a,c) \otimes Hom(c,d) \arrow[r, "\circ"] & Hom(a,d)
\end{tikzcd}$$

- $$\begin{tikzcd}
1\!\!1 \otimes Hom(a,b) \arrow[r, "\lambda"] \arrow[d, "id_a"] & Hom(a,b) & Hom(a,b) \otimes 1\!\!1 \arrow[l, "\rho"'] \arrow[d, "id_b"] \\
Hom(a,a) \otimes Hom(a,b) \arrow[ur, "\circ"] & & Hom(a,b) \otimes Hom(b,b) \arrow[ul, "\circ"']
\end{tikzcd}$$

Ex: * A (Set, ×)-cat. is a loc-small cat

* AbGrp is naturally a AbGrp-cat.

* More generally Mod_R-categories are called R-linear

* The cat. of locally compact topological spaces is naturally a Top-enriched cat. via the compact-open topology on Hom-sets.