Functors:

Def: $\mathcal{C}, \mathcal{D}$ categories.

A functor $F: \mathcal{C} \to \mathcal{D}$ consists of:

* A map $Ob(\mathcal{C}) \to Ob(\mathcal{C})$
$$c \mapsto Fc$$

* $\forall c, c' \in Ob(\mathcal{C})$ a map
$$Hom(c, c') \to Hom(Fc, Fc')$$
$$f \mapsto Ff$$

s.t.: $\cdot$ $\forall c, c', c'' \in Ob(\mathcal{C})$
$\forall f \in Hom(c, c'), f' \in Hom(c', c'')$
$$F(f \circ f') = (Ff) \circ (Ff')$$

$\cdot$ $F(id_c) = id_{Fc}$

Examples:

* U: $\underline{Grp} \to \underline{Set}$ sends a group to the underlying set, a group hom. to the same map of sets.

Same for $Vect_k$, Mod_R, ...

* $P : \mathrm{Set} \to \mathrm{Set}$:
  * $X \in \mathrm{Ob}(\mathrm{Set}) \mapsto PX = \{X' \mid X' \subset X\}$ power set
  * $f : X \to Y \mapsto f_* : PX \to PY$:

    $X' \subset X \mapsto f(X') \subset Y$

* $\pi_0 : \mathrm{Graph} \to \mathrm{Set}$

  G Graph $\mapsto \pi_0(G)$

  $f : G \to G' \mapsto f_* : \pi_0(G) \to \pi_0(G')$

  $[e] \mapsto [f(e)]$

* Euclid_* : Objects: $(n \geq 0$ integer, $x \in \mathbb{R}^n)$

  Morph. $(n, x) \to (m, y)$ are

  diff. maps $f : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(x) = y$

  Functor: $d : \mathrm{Euclid}_* \to \mathrm{Mat}_{\mathbb{R}}$

  $(n, x) \mapsto n$

  $f : (n, x) \to (m, y) \mapsto \left(\frac{\partial f_i}{\partial x_j}(x)\right) \in \mathbb{R}^{m \times n}$

  The fact that this is a functor is equivalent to the chain rule from analysis.

* Can compose functors and have identity functors

  $\to$ Category Cat:

  Objects : Categories

  Morphisms : Functors

These are the basic objects of category theory. So cat. theory studies what we can say about a mathematical object simply from how it relates to other objects.

Some remarks on set theory

Russell: There is no set of all sets!

(Else can form $Y = \{X \text{ set} \mid X \notin X\}$, then $Y \in Y \Leftrightarrow Y \notin Y$ ⚡)

To fix this, we introduce a second kind of collection, so-called classes:

- Sets are "small collections"
- Classes are "large collections"
- Every set is a class, but not vice versa
- Sets can be elements of classes, but classes which are not sets cannot

$\rightsquigarrow \{X \mid X \text{ is a set}\}$ is a class but not a set

This can e.g. be axiomatized by the von Neumann-Bernays-Gödel axioms.

$\rightsquigarrow$ In the def. of a category, we take $Ob(C)$ and the $Hom(c,d)$ to be classes instead of sets.

<u>Def:</u> A cat. C is

* <u>small</u> if $Ob(C)$ and all $Hom(c,d)$ are sets
* <u>locally small</u> if $\forall c,d \in C$: $Hom(c,d)$ is a set.

E.g: The cat. Set is locally small but not small.

Basic notions:

Def: A morphism $c \xrightarrow{f} d$ in a cat. C is an isomorphism if $\exists\, d \xrightarrow{g} c$ s.t.
$g \circ f = id_c$, $f \circ g = id_d$

If such a g exists, it is unique:
If g' is another such arrow, then
$$g = g \circ id_d = g \circ f \circ g' = id_c \circ g' = g'$$
We denote g by f^{-1}.

Ex:
* Isomorphisms in Set are bijections
* Isomorphisms in Grp are group isomorph.
* Isomorphisms in Top are homeomorphisms

Ex: A group is a category with a single object in which every morphism is an isomorphism.

Dual category:

<u>Def:</u> C cat $\rightsquigarrow$ C^{op} cat:

objects: $Ob(C)$

morphisms: $c, d \in C$

$\rightsquigarrow \mathrm{Hom}_{C^{op}}(c,d) := \mathrm{Hom}_C(d,c)$

Composition: $c, d, e \in C$

$$\mathrm{Hom}_{C^{op}}(c,d) \times \mathrm{Hom}_{C^{op}}(d,e) \to \mathrm{Hom}_{C^{op}}(c,e)$$

$$\mathrm{Hom}_C(d,c) \times \mathrm{Hom}(e,d) \to \mathrm{Hom}(e,c)$$

$\rightsquigarrow$ Can check: This is associative and identity morph. in C give id. morph in C^{op}

$\rightsquigarrow C^{op}$ is a cat.

<u>Ex.</u> $(G, \cdot)$ group

$\rightsquigarrow$ new group law: $g_1 * g_2 = g_1 g_2$

$\rightsquigarrow$ If C is the one-object cat. ass. to $(G, \cdot)$ then C^{op} is the one-obj. cat ass to $(G, *)$.

Def: C, D categories

A contravariant functor $C \to D$ is a functor $F: C^{op} \to D$.

i.e this is given by

* $F: Ob(C) \to Ob(D)$

$\forall c \xrightarrow{f} d$ in C, $Fc \xleftarrow{Ff} Fd$

s.t $F(fg) = Fg \cdot Ff$

$F(id_c) = id_{F(c)}$

Ex: * $(\ )^* : Vect_k^{op} \to Vect_k$

$V \mapsto V^* = Hom(V, k)$

$f: V \to W \mapsto f^*: W^* \to V^*$

$h \mapsto h \circ f$

* For a loc. small cat C:

$c \in C \rightsquigarrow$ Functor $C^{op} \to Set$

$d \in C \mapsto Hom(d, c)$

$d \xrightarrow{f} d' \mapsto f^*: Hom(d', c) \to Hom(d, c)$

$h \mapsto h \circ f$