

* coproducts: $(C_i)_{i \in I} \in \mathcal{C}^I$:

The coproduct of the C_i is an object

$\coprod_{i \in I} C_i \in \mathcal{C}$ together with morph.

$$\delta_i: C_i \rightarrow \coprod_{i \in I} C_i \text{ s.t.}$$

$$\forall d \in \mathcal{C}, (f_i: C_i \rightarrow d)_{i \in I} \exists f: \coprod_{i \in I} C_i \rightarrow d$$

$$\text{s.t. } \forall i: f \circ \delta_i = f_i$$

Note: Coproducts in $\mathcal{C}$ are the same as products in $\mathcal{C}^{\text{op}}$!

Ex: Coproducts exist in:

Set: $\coprod C_i$ = disjoint union of the C_i

Ab: C_i ab grp $\bigoplus_{i \in I} C_i$ direct sum

$$C_j \rightarrow \bigoplus_{i \in I} C_i$$

$$g \mapsto (0, \dots, g, \dots, 0)$$

$C_i \xrightarrow{h_i} G$ homomorphisms

$$\sim \bigoplus_{i \in I} C_i \xrightarrow{h} G$$

$$(g_i) \mapsto \sum_{i \in I} h_i(g_i)$$

same eq in Mod $\mathcal{P}$

$\mathcal{C}_{\text{gp}}$: free product of groups

Universal properties

Some examples of universal properties

(Cat:

* products: I (index) set, $(C_i)_{i \in I} \in \text{Ob}(C)^I$

The (direct) product of the C_i is an object $\prod_{i \in I} C_i \in C$ together with morph.

$\pi_i : \prod_{i \in I} C_i \rightarrow C_i$ for all $i \in I$ s.t.:

$\forall d \in C. \forall (f_i : d \rightarrow C_i)_{i \in I} \exists! f : d \rightarrow \prod_{i \in I} C_i$
s.t. $\forall i \in I: \pi_i \circ f = f_i$

These exist e.g. in Set, Mod $_R$, Grp, Top
(product top!), Cat:

$(C_i)_{i \in I}$ categories $\Rightarrow \prod_{i \in I} C_i$ cat:

$$\text{Ob}(\prod_{i \in I} C_i) = \prod_{i \in I} \text{Ob}(C_i)$$

$$\text{Hom}((C_i)_{i \in I}, (C'_i)_{i \in I}) := \prod_{i \in I} \text{Hom}(C_i, C'_i)$$

with termwise composition

* Cat: $(C_i)_{i \in I}$ categories

$\leadsto \coprod_{i \in I} C_i$ category:

$$\text{Ob}\left(\coprod_{i \in I} C_i\right) = \coprod_{i \in I} \text{Ob}(C_i)$$

$$c \in C_i, c' \in C_j \leadsto \text{Hom}(C, C') = \begin{cases} \text{Hom}_{C_i}(C, C') & \text{if } i=j \\ \emptyset & \text{if } i \neq j \end{cases}$$

with the composition from the C_i .

* $C \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C' \text{ in } C:$

an equalizer of f, g is a morphism

$$d \xrightarrow{h} C \text{ s.t. } f \circ h = g \circ h \text{ and s.t.}$$

$$\forall e \xrightarrow{k} C \text{ s.t. } f \circ k = g \circ k$$

$$\exists! e \xrightarrow{e} d \\ h \circ e = k$$

$$\begin{array}{ccc} e & \xrightarrow{k} & C \\ \downarrow h & & \downarrow h \\ d & \xrightarrow{h} & C \end{array} \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C'$$

Eg: In Set: $d = \{x \in C \mid f(x) = g(x)\} \hookrightarrow C$

In Grp: still $d = \{x \in C \mid f(x) = g(x)\}$
which is now a subgroup of C .

Eg: If $g: C \rightarrow C'$ constant $\hookrightarrow$
 $\forall x \in C: x \mapsto 1$

the equalizer of f and g is
the kernel of f .

In cat: $d = \text{Subcat. of } c$

consisting of those objects
 $x \in c$ for which $f(x) = g(x)$

But as we already saw, it is
not a good idea to ask for
equality in a category, so this is
not very useful. We obtain a
more natural construction by
replacing "equality" with "isomorphism":

$C \xrightleftharpoons[F]{F} C'$ functors in cat D :

Objects: $(c, h) : c \in C$

$h: F(c) \rightarrow G(c)$
iso in D

morph

$(c, h) \rightarrow (c', h')$

morph $c \xrightarrow{g} c'$ in C st.

$F(c) \xrightarrow{h} G(c)$

$F(g) \downarrow$

$\downarrow G(g)$

$F(c') \xrightarrow{h'} G(c')$

Commutates

Have functor:

$D \xrightarrow{K} C$

$(c, h) \mapsto c$

$g \mapsto g$

+ an isomorphism of functors

$$F \circ K \cong L \circ K$$

$$(C, h) \mapsto h: F(C) \xrightarrow{\cong} L(C)$$

~ This satisfies a univ. prop.
analogous to the one of an eq.
among all functors $E \xrightarrow{L} C$
together with an isomorphism $F \circ L \cong G \circ L$.

This is the 2-equalizer in the
2-cat. of cat's.

* Dually: A coequalizer of

$$C \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C' \text{ in } C \text{ is a morphism}$$

$C' \xrightarrow{h} D$ s.t. $h \circ f = h \circ g$ and s.t.

$\forall C' \xrightarrow{k} E$ s.t. $h \circ f = h \circ g \Rightarrow \exists ! \delta \xrightarrow{\ell} E : k = \ell \circ h$

$$\begin{array}{ccccc} C & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & C' & \xrightarrow{h} & D \\ & & \downarrow k & & \downarrow \ell \\ & & E & & \end{array}$$

Ex: In Set:

Have eq. rel. $\sim$ on C' given by

$$\forall x, y \in C' \quad x \sim y \Leftrightarrow \exists z \in C : f(z) = x, g(z) = y$$

$$\sim \text{ is } \delta = C' / \sim$$

So eg if $R \subseteq C' \times C'$ is an eq. rel
 then C'/R is the coeq. of $R \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} C'$

In Top: same quotient equipped with quotient topology

In Grp: $C \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C' \rightsquigarrow N = \text{normal subgr.}$

of C gen by $\{f(c)g(c)^{-1} \mid c \in C\}$
 $\leadsto d = C'/N$

So e.g. for $N \subseteq C$ normal,
 the quotient C/N is the
 coequalizer of $N \xrightarrow{\text{incl.}} C$
 $\quad \quad \quad \eta \quad \quad \quad \downarrow$

One can also form 2-coeq.
 of categories, but this is more complicated.

* Initial object: $c \in C$ is an initial
 object if $\forall d \in C \exists! c \rightarrow d$ in C
 (This is the product over the empty set.)

Ex: In Set: $\emptyset$
 In Grp: $\{1\}$
 In Mod $_R$: 0
 In Cat: $\emptyset$

* Dually: A final object $c \in C$ is one st
 $\forall d \in C \exists! d \rightarrow c$ in C

Eg: In Set: $\{*\}$ In Grp: $\{1\}$
 In Mod $_R$: 0 In Cat: $\{*\}$