From these examples we observe:

* These are 2 kinds of univ. properties: Ones where we describe morphisms into the object (e.g. products, coequalizers, final objects), these will be called limits, and ones where we describe morphisms out of the objects (e.g. coproducts, equalizers, initial objects), these will be called colimits.
  The two are exchanged by duality.

So for each univ. prop. of the first kind, if $c \in \mathcal{C}$ is described by a univ. property, we are describing the assignment $d \in \mathcal{C} \mapsto \mathrm{Hom}(d,c)$

In fact, via the unicity part of the univ. prop. we are also describing this functorially in d:

E.g: For $\prod_{i\in I} c_i$:

$F_\pi : C^{op} \to Set$ functor

$$d \mapsto \prod_{i\in I} Hom(d, c_i)$$

$$d \xrightarrow{h} d' \mapsto \prod_{i\in I} Hom(d', c_i) \to \prod_{i\in I} Hom(d, c_i)$$

$$(d'_i \xrightarrow{f_i} c_i) \mapsto (d_i \xrightarrow{h} d'_i \xrightarrow{f_i} c_i)$$

Then if c is a product of the c_i, we have isomorphisms

$$F_\pi(d) \cong Hom(d, c)$$

which are natural in d.

So we start by studying functors $F: C \to Set$ for which such a c exists:

Let C be a locally small cat.

$c \in C \rightsquigarrow Hom(-, c) : C^{op} \to Set$ functor

$$d \in D \mapsto Hom(d, c) \in Set$$

$$d \xrightarrow{h} d' \text{ in } D \mapsto h^*: Hom(d', c) \to Hom(d, c)$$

$$d' \xrightarrow{f} c \mapsto d \xrightarrow{h} d' \xrightarrow{f} c$$

$\mathrm{Hom}(c,-) : C \to \mathrm{Set}$ functor

$d \in D \mapsto \mathrm{Hom}(c,d)$

$d \xrightarrow{h} d'$ in $D \mapsto h_* : \mathrm{Hom}(c,d) \to \mathrm{Hom}(c,d')$

$$\begin{array}{ccc} \mathrm{Hom}(c,d) & \to & \mathrm{Hom}(c,d') \\ \Downarrow & & \Downarrow \\ c \xrightarrow{f} d & \mapsto & c \xrightarrow{f} d \xrightarrow{h} d' \end{array}$$

Def: • A functor $F : C \to \mathrm{Set}$ (resp. $F : C^{op} \to \mathrm{Set}$) is called <u>representable</u> if there exists an object $c \in C$ together with an isomorphism $F \cong \mathrm{Hom}(c,-)$ (resp $F \cong \mathrm{Hom}(-,c)$) of functors.

• Such an isomorphism $F \cong \mathrm{Hom}(c,-)$ or $F \cong \mathrm{Hom}(-,c)$ is called a representation of F.

We will also say that F is represented by c.

By the above, we can think of a representable functor as an encoding of the univ. property of the representing object c.

Examples of representable functors:

* $\mathrm{id}_{Set}: Set \to Set$ is represented by $\{*\} \in Set$:

Have bijections

$$X \cong \mathrm{Hom}_{Set}(\{*\}, X)$$

$$h(*) \leftarrow\!\shortmid \{*\} \xrightarrow{h} X$$

which are natural in X:

$\forall\, X \to Y$ in Set

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[r, phantom, "\cong"] & \mathrm{Hom}(\{*\}, X) \arrow[d, "f_*"] & h(*) \arrow[d, phantom, "\mathrm{I}"] & h \arrow[l, mapsto] \arrow[d, phantom, "\mathrm{I}"] \\ Y \arrow[r, phantom, "\cong"] & \mathrm{Hom}(\{*\}, Y) & f(h(*)) & f\circ h \arrow[l, mapsto] \end{tikzcd}$$

* The forgetful functor $U: Grp \to Set$ is represented by $\mathbb{Z} \in Grp$:

$$G \in Grp \rightsquigarrow \quad U(G) \cong \mathrm{Hom}_{Grp}(\mathbb{Z}, G)$$

$$h(1) \leftarrow\!\shortmid h \in \mathrm{Hom}_{Grp}(\mathbb{Z}, G)$$

and this is again natural in G.

* R ring : $U : \mathrm{Mod}_R \to \mathrm{Set}$
  forgetful functor

  is representable by $R \in \mathrm{Mod}_R$:

  $$M \in \mathrm{Mod}_R \rightsquigarrow U(M) \cong \mathrm{Hom}_{\mathrm{Mod}_R}(R, M)$$

  $$h(1) \leftarrow\!\shortmid h$$

* $U : \mathrm{Ring} \to \mathrm{Set}$ forgetful functor
  is representable by the polynomial
  ring $\mathbb{Z}[X]$:

  $$R \in \mathrm{Ring} : U(R) \cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[X], R)$$

  $$h(X) \leftarrow\!\shortmid h$$

The inverse of this map sends $r \in R$
to the ring homomorphism

$$h : \mathbb{Z}[X] \to R$$

$$f = \sum_{i=0}^{n} a_i X^i \mapsto f(r) = \sum_{i=0}^{n} a_i r^i$$

* We can extend this example to polynomial equations: Consider e.g. the functor

$$X : \mathrm{Ring} \to \mathrm{Set}$$
$$R \mapsto \{(r_1, r_2, r_3) \in R \mid r_1^3 + r_2^3 = r_3^3\}$$
$$R \xrightarrow{h} R' \mapsto X(R) \to X(R')$$
$$(r_1, r_2, r_3) \mapsto (h(r_1), h(r_2), h(r_3))$$

The functor X is representable by the ring $Q = \mathbb{Z}[X_1, X_2, X_3] / (X_1^3 + X_2^3 - X_3^3)$:

$$R \in \mathrm{Ring} \rightsquigarrow X(R) \cong \mathrm{Hom}(Q, R) \qquad \overline{X}_1, \overline{X}_2, \overline{X}_3 \in Q$$
$$(h(\overline{X}_1), h(\overline{X}_2), h(\overline{X}_3)) \leftarrow\!\shortmid h$$

Inverse: $(r_1, r_2, r_3) \in X(R)$

$$\mapsto h : Q \to R$$
$$f + (X_1^3 + X_2^3 - X_3^3) \mapsto f(r_1, r_2, r_3),$$

this is well-defined since $(r_1, r_2, r_3) \in X(R)$.

The same works for any equation $P(X_1, \ldots, X_n) = 0$ with $P \in \mathbb{Z}[X_1, \ldots, X_n]$.

Some contravariant examples:

* Functor $\mathcal{O}: \mathrm{Top}^{op} \to \mathrm{Set}$

$$X \mapsto \{U \subset X \mid U \text{ open in } X\}$$

$$X \xrightarrow{f} Y \text{ cont} \mapsto \mathcal{O}(Y) \to \mathcal{O}(X)$$

$$U \subset Y \mapsto f^{-1}(U) \subset X$$

This functor is representable by the Sierpinski space S:

$S = \{a, b\}$ with open subsets $\emptyset, \{a,b\}, \{a\}$

For $X \in \mathrm{Top}$ there are mutually inverse bijections which are natural in X:

$$U \subset X \mapsto \begin{array}{l} h: X \to S \\ x \mapsto \begin{cases} a & x \in U \\ b & x \notin U \end{cases} \end{array}$$

$$\mathcal{O}(X) \rightleftarrows \mathrm{Hom}_{\mathrm{Top}}(X, S)$$

$$h^{-1}(\{a\}) \leftarrow h$$

* $A, B \in \mathrm{Set}$

$\rightsquigarrow$ functor $\mathrm{Hom}(\_ \times A, B): \mathrm{Set}^{op} \to \mathrm{Set}$

$$X \mapsto \mathrm{Hom}(X \times A, B)$$

$$f: X \to X' \mapsto \mathrm{Hom}(X' \times A, B) \to \mathrm{Hom}(X \times A, B)$$

$$X' \times A \xrightarrow{h} B \mapsto X \times A \xrightarrow{f \times id_A} X' \times A \to B$$

This functor is representable by the set $B^A \doteq Hom_{Set}(A, B)$

$$X \in Set \rightsquigarrow Hom(X \times A, B) \cong Hom(X, B^A)$$

$$X \times A \to B \quad \leftarrow \quad h : X \to B^A$$

$$(x, a) \mapsto (h(x))(a)$$

This is called currying in Computer Science.