Yoneda Lemma

Given a functor $C \to Set$, what is needed to define a nat. trans.

$$Hom(c, -) \Rightarrow F$$

for some $c \in C$?

More generally, we can describe nat. transf.

$$Hom(c, -) \to F$$

as follows:

Given $F: C \to Set$ and $c \in C$ there is a map

$$\begin{aligned} \Phi : Hom(Hom(c,-), F) &\to F \\ \alpha &\mapsto \alpha_c(id_c) \end{aligned}$$

Thm (Yoneda lemma)

The map Φ is a bijection

Pf: Bijectivity:

We construct an inverse map

$\Psi : F(c) \to \ldots$

as follows:

Let $x \in Fc$: For $d \in C$ need $\psi(x)_d$:

$$\mathrm{Hom}(c,d) \xrightarrow[\psi(x)_d]{} Fd$$

For $c \xrightarrow{f} d$ in C, we look at

$$\begin{tikzcd} \mathrm{Hom}(c,c) \arrow[r, "\psi(x)_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\ \mathrm{Hom}(c,d) \arrow[r, "\psi(x)_d"'] & Fd \end{tikzcd}$$

Since $f = f \circ id_c = f_*(id_c)$, for this to commute we need

$$\psi(x)_d(f) = \psi(x)_d(f_*(id_c))$$

$$= Ff(\underbrace{\psi(x)_c(id_c)}_{=x}) = Ff(x)$$

if ψ is inverse to Φ

So we define $\psi(x)_d(f) := Ff(x)$.

We need to check that this defines a natural transformation

$$\psi(x) : \mathrm{Hom}(c,-) \to F$$

So for some $g: d \to e$ in C we need the following square to commute:

$$\begin{tikzcd}
\mathrm{Hom}(c,d) \arrow[r, "\psi(x)_d"'] \arrow[d, "g_*"'] & Fd \arrow[d, "Fg"] \\
\mathrm{Hom}(c,e) \arrow[r, "\psi(x)_e"'] & Fe
\end{tikzcd}$$

For $f \in \mathrm{Hom}(c,d)$ we check:

$$\psi(x)_e(g_*(f)) = \psi(x)_e(g \circ f) = F(g \circ f)(x)$$

$$Fg(\psi(x)_d(f)) = Fg(Ff(x))$$

These are equal since F is a functor.

So we have defined ψ.

We still need to check that ψ is inverse to Φ:

Let $\alpha \in \mathrm{Hom}(\mathrm{Hom}(c,-), F)$, $f: c \to d \in C$:

$$\psi(\Phi(\alpha))_d(f) = \psi(\alpha_c(id_c))_d(f) = Ff(\alpha_c(id_c))$$
$$= \alpha_d(f_*(id_c)) = \alpha_d(f)$$
$$\Rightarrow \quad \psi(\Phi(\alpha)) = \alpha$$

Let $x \in Fc$:

$$\Phi(\psi(x)) = \psi(x)_c(id_c) = F(id_c)(x) = id_c(x) = x$$

$\square$

Prop: The map Φ is natural in F and c:

Pf: Naturality in c:

Let $c \xrightarrow{h} c'$ in C. This induces a nat. transformation

$$\beta^h : \mathrm{Hom}(c',-) \to \mathrm{Hom}(c,-)$$
$$d \mapsto \mathrm{Hom}(c',d) \xrightarrow[h^*]{} \mathrm{Hom}(c,d)$$

Then the following diagram commutes:

$$\begin{tikzcd}
\alpha \in \mathrm{Hom}(\mathrm{Hom}(c,-),F) \arrow[r, "\Phi"] \arrow[d] & Fc \arrow[d, "Fh"] \\
\mathrm{Hom}(\mathrm{Hom}(c',-),F) \arrow[r, "\Phi"'] & Fc'
\end{tikzcd}$$

$$\alpha \mapsto \alpha \circ \beta^h$$

Here: $Fh(\Phi(\alpha)) = Fh(\alpha_c(id_c))$

$$\Phi(\alpha \circ \beta_h) = \alpha_{c'}(\beta^h_{c'}(id_{c'})) = \alpha_{c'}(h)$$

These are equal since α is a nat. transf.

Naturality in F: $F': C \to Set$

$\gamma : F \to F'$ nat. transf.

Then

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{Hom}(c,-), F) \arrow[r, "\Phi"] \arrow[d, "\alpha \mapsto \gamma\circ\alpha"'] & Fc \arrow[d, "\gamma_c"] \\ \mathrm{Hom}(\mathrm{Hom}(c,-), F') \arrow[r] & F'c \end{tikzcd}$$

commutes:

$$\gamma_c(\Phi(\alpha)) = \gamma_c(\alpha_c(id_c))$$

$$\Phi(\gamma\circ\alpha) = (\gamma\circ\alpha)_c(id_c) = \gamma_c(\alpha_c(id_c))$$

by the composition rule for nat. transf.

$\square$

A first application:

<u>Cor:</u> $c, d \in C \leadsto$

$$\mathrm{Hom}(c,d) \to \mathrm{Hom}(\mathrm{Hom}(d,-), \mathrm{Hom}(c,-))$$

$$f \mapsto f^*$$

is a bijection.

<u>Pf:</u> Take $F = \mathrm{Hom}(c,-)$ in the Yoneda lemma. This gives a

bijection

$$\Phi : \mathrm{Hom}(\mathrm{Hom}(d,-), \mathrm{Hom}(c,-)) \to \mathrm{Hom}(c,d)$$
$$\alpha \mapsto \alpha(\mathrm{id}_d)$$

For $f: c \to d$, this sends $\alpha = f^*$
to $f^*(\mathrm{id}_c) = f \circ \mathrm{id}_c = f$

$\Rightarrow f \mapsto f^*$ is inj.

Since Φ is inj, any $\alpha : \mathrm{Hom}(d,-) \to \mathrm{Hom}(c,-)$
is equal to $(\alpha(\mathrm{id}_d))^*$.

$\Rightarrow f \mapsto f^*$ is surj.

$\square$

Cor: For $c, d \in \mathcal{C}$ if the functors $\mathrm{Hom}(c,-)$ and $\mathrm{Hom}(d,-)$ are isomorphic, so are c and d.

So if we know all morphisms out of an object (or dually all morph. into an object) we know the object (up to iso.)!
In particular, a representation $\mathrm{Hom}(c,-) \cong F$ of a functor F is unique up to isomorphisms $c \cong c'$.

Up to size issues, all of this can be formulated more concisely as follows:

Assume that C is small. Then there is a functor category $\mathrm{Fun}(C, \mathrm{Set})$ and a functor

$$y: C^{op} \to \mathrm{Fun}(C, \mathrm{Set})$$
$$c \mapsto \mathrm{Hom}(c, -)$$
$$f: c \to d \mapsto f^*$$

The first cor. says that y is fully faithful.

Furthermore, the Yoneda lemma can be expressed as an isom. of functors

Have 2 functors

$$C \times \mathrm{Fun}(C, \mathrm{Set}) \to \mathrm{Set}$$

$\mathrm{Hom}(y(-), -) : (c, F) \mapsto \mathrm{Hom}(\mathrm{Hom}(c, -), F)$

which acts on morph. as in the functoriality statement after the Yoneda lemma

and $ev : (c, F) \mapsto F(c)$

$$(c \xrightarrow{f} c', F \xrightarrow{\gamma} F') \mapsto F(c) \xrightarrow{Ff} F(c') \xrightarrow{\gamma_{c'}} F'(c')$$

and the Φ give an isomorph. of func.

$$\mathrm{Hom}(Y(-),-) \to \mathrm{ev}.$$

By applying the above to C^{op} we get dual statements:

<u>Thm</u> Let $F: C^{op} \to \mathrm{Set}$ be a functor and $c \in C$.

Then $$\Phi: \mathrm{Hom}(\mathrm{Hom}(-,c), F) \to Fc$$
$$\alpha \mapsto \alpha_c(\mathrm{id}_c)$$

is a bijection which is natural in F and c.

<u>Cor</u>: For $c, d \in C$ the map

$$\mathrm{Hom}(c,d) \to \mathrm{Hom}(\mathrm{Hom}(-,c), \mathrm{Hom}(-,d))$$
$$f \mapsto f_*$$

is a bijection.

Another application:

In linear algebra, we have "row operations" on the set of matrices with n rows with coefficients in some ring R

(exchanging two rows, adding a multiple of one row to another, multiplying a row by an element of $R^\times$)

<u>Cor</u>: Every row operation is defined by left mult. by some $n \times n$-matrix, which is obtained by applying the row operation to the identity matrix.

Pf: We consider the category Mat_R and for $n \geq 0$ the functor

$$\mathrm{Hom}(\_, n) : \mathrm{Mat}_R^{op} \to \mathrm{Set},$$

this sends $m \geq 0$ to $R^{m \times n}$.

The fact that matrix mult. is linear implies that each row operation defines a nat. transf.

$\mathrm{Hom}(-,n) \to \mathrm{Hom}(-,n)$

Hence by the corr. the row operation is defined by the element of $\mathrm{Hom}(n,n)$ obtained by applying this nat. transf. to id_n. $\square$

<u>Def.</u> A universal property of $c \in C$ is a functor $F: C \to Set$ or $C^{op} \to Set$ together with an element $x \in Fc$ which induces via the Yoneda lemma an isomorphism

$F \cong \mathrm{Hom}(c,-)$ or $F \cong \mathrm{Hom}(-,c)$