Limits and Colimits

In our examples of univ. properties, we saw the following pattern:
We have a "diagram" in C consisting of some objects $(c_i)_{i \in I}$ and some morphisms $c_i \to c_j$

(e.g. just the $(c_i)_{i \in I}$ for products, $c \rightrightarrows d$ for equalizers)

and we look for a universal object $c \in C$ with maps $c \to c_i$ for all $i \in I$ s.t for all $c_i \to c_j$ in our diagram

$$\begin{tikzcd} c \arrow[r] \arrow[dr] & c_i \arrow[d] \\ & c_j \end{tikzcd}$$

commutes.

<u>Def:</u> Let I be a category.

* A diagram in C of shape I, or indexed by I, is a functor $I \to C$.

* For $c \in C$, the constant functor $c : I \to C$ sends every $i \in I$ to c and every morphism in I to id_c

* For $F: I \to C$ and $c \in C$, a cone over F with apex c is a nat. transf. $c \to F$.

(So such a cone amounts to giving morph $c \to F_i$ for all $i \in I$ s.t. for all $i \xrightarrow{f} j$ in I the diagram

$$\begin{tikzcd} & c \arrow[dl] \arrow[d] \\ F_i \arrow[r] & F_j \end{tikzcd}$$

commutes.)

<u>Def</u>: Let $F: I \to C$ be a functor.

* $\mathrm{Cone}(-, F) : C^{op} \to \mathrm{Set}$ functor

$c \in C \mapsto \{\text{cones } c \to F\}$

$c \xrightarrow{f} d$ in C $\rightsquigarrow$ nat. transf $c \xrightarrow{\alpha_f} d$ of constant cones

$\rightsquigarrow \{\text{cones } d \to F\} \to \{\text{cones } c \to F\}$

$\alpha \mapsto \alpha \circ \alpha_f$

* A limit of F is a representation of $\mathrm{Cone}(-, F)$, that is an object $c \in C$ together with an element $\lambda \in \mathrm{Cone}(c, F)$ which induces an iso

$\mathrm{Hom}(-,c) \cong \mathrm{Cone}(-,F)$

* E.g.: I index set, which we consider as a discrete cat., that is with only the identity morph, then a functor $F: I \to C$ amounts to choosing objects $c_i = F(i)$ for all $i \in I$, and a limit of F is the same as a product of the c_i.

Similarly, equalizers are limits over the category $\bullet \rightrightarrows \bullet$ with 2 objects and 4 morphisms

<u>Prop</u>: Let $(c, \lambda: c \to F)$ and $(c', \lambda': c' \to F)$ be limits of F. Then there exists a unique isomorphism $c \cong c'$ s.t. for all $i \in I$ the diagram

$$\begin{tikzcd} c \arrow[r, "\lambda_i"] & Fi \\ c' \arrow[u, "\cong"] \arrow[ur, "\lambda'_i"'] & \end{tikzcd}$$

commutes.

Pf: Have isomorphisms

$\mathrm{Hom}(-,c) \cong \mathrm{Cone}(-,F) \cong \mathrm{Hom}(-,c')$,

by Yoneda this comes from an isomorphism $c \cong c'$ with the required property.

$\square$

Hence, if a limit of F exists, we choose one and call it "the" limit $\lim_I F$.

Dually we can define colimits:

Def: Given $c \in C$ and $F: I \to C$, a cone under F with nadir c is a nat. transf. $F \to c$.

- $\rightsquigarrow$ Functor $\mathrm{Cone}(F,-): C \to \mathrm{Set}$

$$c \in C \mapsto \mathrm{Cone}(F,c)$$
$$c \xrightarrow{f} d \mapsto \mathrm{Cone}(F,c) \to \mathrm{Cone}(F,d)$$
$$F \xrightarrow{\lambda} c \mapsto F \xrightarrow{\lambda} c \xrightarrow{f_*} d$$

- A colimit of F is an object $c \in C$ together with $\lambda: F \to c$ inducing an isomorphism $\mathrm{Cone}(F,-) \cong \mathrm{Hom}(c,-)$ of functors.

If they exist, colimits are unique as above and we speak of "the" colimit $\mathrm{colim}_I F$.

So e.g. coproducts are colimits indexed by discrete categories and coequalizers are colimits indexed by $\cdot \rightrightarrows \cdot$.

Usually, one can only expect limits/colimits indexed by small categories to exist. We call these small (co)limits.
E.g. let $c, d \in C$ with two different morphisms $c \rightrightarrows d$. If $\prod_{i \in I} d$ exists for some class I, then there are 2^I many morphisms $c \to \prod_{i \in I} d$.
So if I is e.g. larger than
$$\mathrm{Hom}(C) = \coprod_{c, d \in C} \mathrm{Hom}(c, d),$$
this product can't exist for size reasons.

In general, limits / colimits of shape I need not exist in a cat. C. If they do, we say that C admits limits / colimits of shape I.

Def: A category C is complete (resp. cocomplete) if it admits all small limits (resp. colimits).