Category Theory
Exercise Sheet 11 - Solutions

Exercise 1. Show that the forgetful functor U from the category of fields to the category of rings admits neither a left nor a right adjoint.

Solution. The category of rings has an initial object (namely the ring $\mathbb{Z}$), whereas the category of fields does not (e.g. since there are no homomorphisms between fields of different characteristic). If there were such a left adjoint, it would preserve all small colimits, and so in particular initial objects. Since this is not possible, there is no such left adjoint.

If there exists a right adjoint R, then for every ring A, the identity $R(A) \to R(A)$ corresponds to a ring homomorphism $U(R(A)) \to A$. But e.g. for $A = \mathbb{Z}$, there exists no homomorphism $U(F) \to A$ from a field F.

Exercise 2. Use the adjoint functor theorem to prove the existence of free groups.

Solution. The free group functor is left adjoint to the forgetful functor U from the category of groups to the category of sets. We already know that U is continuous, so we need to verify the solution set condition. So for a set S, we need to look at the class of maps $\{S \to U(G) \mid G \in \mathrm{Grp}\}$. For such a map $f\colon S \to U(G)$, let G' be the subgroup of G generated by the image of S. Every element of G' is a finite products of elements $f(s)^{\pm}$ for some $s \in S$. Hence the cardinality of G' is bounded above by the cardinality of $\prod_{i\geq 0}(2^S)^i$. The groups G' satisfying such a bound form a set, and hence the functor U satisfies the solution set condition.

Exercise 3. Let C be a locally small category which admits all small coproducts. Show that a functor $F\colon C \to \mathrm{Set}$ is representable if and only if it admits a left adjoint.

Lösung. If F admits a left adjoint functor L, the natural isomorphisms

$$F(c) \cong \mathrm{Hom}(\{*\}, F(c)) \cong \mathrm{Hom}(L\{*\}, c)$$

show that $F(c)$ is representable by $L\{*\}$.

Conversely, if F is representable by $c \in C$, we construct a left adjoint L as follows: We write each set S as $S = \coprod_{s\in S}\{s\}$ and define

$$LS = \coprod_{s\in S} c.$$

Similary, any map $f\colon S \to S'$ we send to the induced morphism

$$Lf = \coprod_{s\in S} c \to \coprod_{s\in S'} c.$$

(The fact that a left adjoint preserves coproducts, and the above fact that $L(\{*\}) = c$ in case a left adjoint L exists, imply that L must have this form if it exists.)

For any set S and object $d \in C$, the universal property of the coproduct gives a natural bijection

$$\mathrm{Hom}(LS, d) \cong \mathrm{Hom}(c, d)^S.$$

Using the natural bijections

$$\mathrm{Hom}(c, d)^S \cong Fd^S = \mathrm{Hom}(S, Fd)$$

we find the desired adjunction. □