Category Theory
Exercise Sheet 9 - Solutions

Exercise 1. For a group G and a functor $F\colon BG \to Set$, describe $\lim_{BG} F$ and $\operatorname{colim}_{BG} F$.

Solution. By a previous exercise, giving a functor $F\colon BG \to Set$ amounts to giving a set $X = F(*)$ together with a left G-action on X.

Since $\lim_{BG} F \cong \operatorname{Hom}(\{*\}, \lim_{BG} F) \cong \operatorname{Cone}(\{*\}, F)$, under this identification of functors $BG \to \mathrm{Set}$ with G-sets, $\lim_{BG} F$ is the set of G-equivariant maps $\{*\} \to X$, where $\{*\}$ is equipped with the trivial G-action. Hence $\lim_{BG} F$ is the set of points of X which are fixed by G.

For a set S, a cone $F \to S$ is the same as a map $X = F(*) \to S$ which is equivariant with respect to the G-action on S leaving all of S fixed. From this it follows that $\operatorname{colim}_{BG} F$ is the quotient of X by the smallest equivalence relation $\sim$ for which $g \cdot x \sim x$ for all $g \in G$ and $x \in X$. (This quotient of X is called the coinvariants of the G-action.)

Exercise 2. For a small category C, show that $\operatorname{Fun}(C, \mathrm{Set})$ is complete and cocomplete and describe small (co)limits in this category.

Solution. (Co)limits in $\operatorname{Fun}(C, \mathrm{Set})$ can be formed "termwise": Let I be a small category and $F\colon I \to \operatorname{Fun}(C, \mathrm{Set})$ a diagram. For each $c \in C$, there is a functor $\mathrm{ev}_c\colon \operatorname{Fun}(C, \mathrm{Set}) \to \mathrm{Set}$ given by evaluation at c. By composition we obtain a diagram $\mathrm{ev}_c \circ F\colon I \to \mathrm{Set}$ whose limit we denoty by L_c. By the functoriality of limits, these objects L_c are naturally functorial in c and so we obtain a functor $L\colon I \to \mathrm{Set}$ together with a cone $F \to L$. The fact that each L_c is a limit of $\mathrm{ev}_c \circ F$ implies that L is a limit of F.

The same construction applies to colimits.

Exercise 3. Let C be a locally small category and $c \in C$. Show that the functor $\operatorname{Hom}(c, \_)\colon C \to \mathrm{Set}$ preserves all limits which exist in C.

Solution. Let $F\colon I \to C$ be a diagram which admits a limit in C. By the definition of limits, a morphism $c \to \lim_I F$ is the same as a cone $\lambda\colon c \to F$. But this is the same as morphisms $\lambda_i\colon c \to Fi$ for all $i \in I$ such that for all $f\colon i \to j$ in I the morphism λ_j is equal to $Ff \circ \lambda_i$. Hence by the description of limits in Set from the last exercise sheet, the set of such λ is equal to the limit of the functor $\operatorname{Hom}(c, \_) \circ F\colon I \to \mathrm{Set}$. This proves the claim.

Exercise 4. Show that every group can be written as a colimit of a diagram consisting of finitely generated groups.

Lösung. Let G be a group and consider the category I whose objects are the finitely generated subgroups H of G, and whose morphisms are the inclusions of such subgroups. This category comes with a natural inclusion functor $F\colon I \to \mathrm{Grp}$. The inclusions $H \to G$ induce a homomorphism $\operatorname{colim}_I F \to G$. We claim that this is an isommorphism. Indeed, consider a group G' together with a cone $F \to G'$. Such a cone amounts to giving homomorphisms $H \to G'$ for all finitely generated subgroups H of G which are compatible with inclusions between such subgroups. Since G is the union of all such H, such homomorphisms fit together uniquely to a homomorphism $G \to G'$. This proves the required universal property. □