Gradient Flows Face-to-Face 4

Raitenhaslach, 9 – 12 September 2024

https://www.math.cit.tum.de/math/forschung/gruppen/analysis-and-mathbio/events/gradient-flows-face-to-face-4/

Program

Day	Monday 09.09	Tuesday 10.09	Wednesday 11.09	Thursday 12.09
08:30	Shuttle from Hotel Glöcklhofer to Raitenhaslach			
09:00–09:40	Peletier	Mielke	Santambrogio	Baradat
09:45–10:25	Léonard	Stephan	Hraivoronska	Hoeksema
10:30–11:00	**Coffee Break**			
11:00–11:40	Di Francesco	Lisini	Caillet	Lunch
11:45–12:25	Radici	Pietschmann	Cancès	
12:30–13:30	**Lunch**			
13:30–15:30	**Discussion** & **Tea Break**			
15:30–16:10	Bruna	Fagioli	Iorio	
16:15–16:55	Portinale	Tanaka	David	
17:15	Shuttle from Raitenhaslach to Hotel Glöcklhofer			

Welcome Address/Practical information

- The meeting takes place at the Raitenhaslach conference centre.
- Accommodation: Hotel Glöcklhofer Ludwigsberg 4, 84489 Burghausen, phone +49 8677 916 400.
- There is a shuttle service between the hotel and Raitenhaslach, leaving **8:30** at the hotel and returning **17:15** from Raitenhaslach.
- Meals:
  - Lunch will be talken in Raitenhaslach.
  - On Monday there will be a buffet dinner at Hotel Glöcklhofer at 18:30.
  - A conference dinner will be held at Strizzi, Stadtplatz 111, Burghausen on Tuesday, 18:30 pm.
- The meeting finishes with lunch on Thursday.
- Shuttles for Thursday to return from Raitenhaslach will be organised during the meeting.

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List of Abstracts

Entropic JKO scheme for the Muskat problem

Aymeric Baradat
CNRS & Université Claude Bernard Lyon 1

The Muskat problem is a system of PDEs describing the evolution of two incompressible and immiscible fluids of different densities under the action of gravitation. When the heavier fluid stands above the lighter fluid, the so-called Saffman-Taylor instability makes the system ill-posed in Sobolev spaces. Yet, Otto proposed in the late 90s to study a relaxation of this problem by interpreting it as a Wasserstein gradient flow and then by considering the limit of the corresponding JKO scheme as the time-step goes to zero. This procedure allowed him to draw a link between the Muskat problem and the entropic solutions of one-dimensional conservation laws. In this talk, I will show that an entropic version of the JKO scheme is particularly well adapted to this problem. On the practical side, it can be efficiently computed using a very simple Sinkhorn algorithm. On the theoretical side, it converges towards the solution of a well-posed system, shedding light on Otto's connection with conservation laws. This is a work in progress with Sofiane Cherf.

Not-a-gradient-flow: funny things that non-equilibrium systems may do

Maria Bruna
University of Oxford

In this talk, I will discuss a model mixture of active (self-propelled) and passive (diffusive) particles with non-reciprocal effective interactions (or forces that violate Newton's third law). We derive the hydrodynamic PDE limit for the particle densities, which is not a Wasserstein gradient flow of any free energy, consistent with the microscopic model having non-equilibrium steady states. We study the emergence of collective behaviour, which includes phase separation and dynamical (travelling) steady states.

Finite elements for p-Wasserstein gradient flows

Clément Cancès

INRIA Lille

W_p-Wasserstein gradient flows yield degenerate parabolic equations involving a q-Laplacian type operator, with q being p's conjugate exponent. Specific difficulties occur when p is chosen not equal to 2. It is in particular required to compute the whole gradient of the driving potential to compute the fluxes, making usual strategies based on two-point flux approximation finite volumes irrelevant. We propose a finite element scheme building on conformal lowest order elements with mass lumping and a backward Euler time discretization strategy. Our scheme preserves mass and positivity while energy decays in time. Its convergence can furthermore be established thanks to material from the theory of gradient flows in metric spaces by Ambrosio, Gigli & Savaré. The analytical results are confirmed by numerical simulations.

This is joint work with D. Matthes, F. Nabet and E.-M. Rott.

Existence for Doubly Nonlinear PDEs via p-Wasserstein gradient flows

Thibault Caillet

Université Claude Bernard - Lyon 1

I will present the results of a recent paper proving a general existence result for a class of Doubly Nonlinear PDEs using p-Wasserstein gradient flows. Classically, one can use the p-JKO scheme and derive a p,q-EDI condition for the limit curve that, along with a suitable chain rule, proves the curve is a solution of the PDE. This works well within the setting of geodesically convex driving energies, and I will show how the same principles can be used to prove the existence of solutions to the PDE without this assumption. Crucially, we will use the fact that a convex function can approximately be written as the difference of two convex functions satisfying McCann's condition, for which one can prove suitable chain rules under technical integrability conditions which can be obtained using the flow-interchange technique.

Inviscid and incompressible limits for tissue growth models: from Brinkman's law to the Hele-Shaw problem

Noemi David

Université Claude Bernard - Lyon 1

Cancer growth modelling has seen an increasing application of fluid-dynamics concepts to describe the mechanical properties of living tissue. The biomechanical pressure plays a central role, both as the driving force of cell movement and as an inhibitor of cell proliferation. Singular limits that can build a bridge between models with different pressure-velocity or pressure-density relations have attracted great interest in recent years. In particular, the theory on the inviscid limit from a visco-elastic model to porous-medium-like equations and the incompressible limit that links the latter to a Hele-Shaw problem with density constraint is nowadays quite well understood. In this talk, I will address the question of passing to the joint limit from a Brinkman compressible model to the Hele-Shaw free boundary problem. To this end, we exploit the gradient flow structure of the limit problem and a family of energy inequalities satisfied by the Brinkman model in order to prove the strong compactness of the velocity field. This is a work in progress in collaboration with Matt Jacobs and Inwon Kim.

Deterministic particle methods for nonlinear transport PDEs

Marco Di Francesco

Università degli Studi dell'Aquila

We will review recent advances on the approximation of solutions to various nonlinear transport PDEs via deterministic particles. The method applies to models which may be written as a continuity equation in which the velocity field may depend (in various ways) from the unknown density. The "particle" equation is then obtained via "Lagrangian" discretisation of the velocity field and via discrete reconstruction of the density. Such a method has been used both as a "validation" of continuum models via discrete ones, and to solve certain PDEs of transport type. Many results have been obtained in the last ten years circa. This talk is an attempt to review them, to emphasize links with other mathematical tools (such as optimal transport), and to try to outline possible new directions.

Small inertia limit for coupled kinetic swarming models

Simone Fagioli

Università degli Studi dell'Aquila

We investigate various versions of multi-dimensional systems involving many species, modeling aggregation phenomena through nonlocal interaction terms. We establish a rigorous connection between kinetic and macroscopic descriptions by considering the small-inertia limit at the kinetic level. The results are proven either under smoothness assumptions on all interaction kernels or under singular assumptions for *self-interaction* potentials. Utilizing different techniques in the two cases, we demonstrate the existence of a solution to the kinetic system, provide uniform estimates with respect to the inertia parameter, and show convergence towards the corresponding macroscopic system as the inertia approaches zero.

This is joint work with Y-.P. Choi and V. Iorio.

Gradient flow formulations for singular jump processes in metric spaces

Jasper Hoeksema

TU Eindhoven

In this talk we discuss a generalisation of variational formulations for the flow corresponding to jump processes with a singular kernel, to the setting of metric spaces. Similar to the local case, this problem is well-posed if the densities of the laws with respect to a reversible invariant measure are bounded from above and above, and a certain Lipschitz density result holds. We will give various examples and discuss cases where this type of approach fails.

Joint work with Riccarda Rossi and Oliver Tse.

The fully discrete JKO scheme for nonlinear diffusion and crowd motion models

Anastasiia Hraivoronska

Université Claude Bernard - Lyon 1

This talk presents a formulation of the JKO scheme restricted to atomic measures on the regular grid as a discrete-in-space approximation to the standard JKO scheme. We discuss the application of this fully discrete formulation for developing new numerical schemes for nonlinear diffusion equations with drift and the crowd motion model. The main result of this presentation is the convergence of the scheme to the corresponding PDEs as the time and space discretization vanishes.

On deterministic particle approximations of continuity equation with Morse-type nonlocal interactions

Valeria Iorio

Università dell'Aquila

In this talk, we study deterministic particle methods for continuity equation with nonlocal interactions driven by Morse-type potentials. We work in the set $\mathcal{P}_2(\mathbb{R})$ of probability measures on $\mathbb{R}$ with finite second moment equipped with the 2-Wasserstein distance. Firstly, the one-species case with the repulsive Morse potential is detailed. We prove that the particle scheme converges towards weak solutions to the nonlocal equations as the number of particles goes to infinity. Furthermore, since the Morse potentials is rescaled to approach a Dirac delta, the scheme approximates the quadratic porous medium equation.

Afterwards, we study a system of two continuity equations with both attractive and repulsive Morse potentials. Once existence, uniqueness and stability result is provided in $\mathcal{P}_2(\mathbb{R})$ via JKO scheme and gradient flow solutions, for the proposed deterministic particle scheme we prove that the discrete densities related to the scheme converge towards the gradient flow solution as the number of particles increases to infinity.

A bridge to the early universe

Christian Léonard

Université Paris Nanterre

Twenty years ago, Yann Brenier, Uriel Frisch, Roya Mohayaee and some coauthors designed an algorithm based on optimal transport for reconstructing the early universe. Meanwhile, in a series of papers Yann Brenier proposed a modified theory of gravitation also based on optimal transport (Monge-Ampère gravitation) and built a related system of Brownian particles experiencing large deviations from some gradient flow evolution, as a bridge between early and today discretized universes.

It turns out that the least action principle attached to Brenier's model is connected to an entropic transport problem (a.k.a. Schrödinger problem) for the empirical measure of Brownian particles experiencing a mean field quantum interaction.

This a joint work with Roya Mohayaee (Institut d'astrophysique de Paris).

Existence of gradient flow solutions for fractional thin film equations with aggregation on convex domains

Stefano Lisini

Università degli Studi di Pavia

I will show a gradient flow structure for a family of fractional thin film equations with linear mobility and a reaction term. The problem is posed in a bounded convex domain with the homogeneous Neumann boundary condition. Existence and properties of weak solutions will be illustrated.

On the stability of NESS in gradient systems with ports

Alexander Mielke
WIAS Berlin

In gradient flows, the steady states are given by the critical points of the driving functional. Hence, the associated fluxes vanish. If a gradient system is coupled to the environment via so-called ports, then steady states may have non-zero fluxes and are called Non-Equilibrium Steady States (NESS). We consider so-called port gradient systems, where boundary conditions act as constraint which generate suitable (boundary) fluxes as Lagrange multipliers.

Such systems may arise on the fast time scale in the limit of slow-fast gradient systems, which leads to the general question under what conditions a port gradient system has a unique NESS and whether it is globally stable. We discuss this question using a novel saddle-point formulation for NESS and relate the results to Prigogine's dissipation principle of 1947. The latter states that under suitable conditions, NESS are minimizers of the dissipation; but in general cases this is false.

A. Mielke: Non-equilibrium steady states as saddle points and EDP-convergence for slow-fast gradient systems. J. Math. Physics 64:12 (2023) 1-20.

Why does entropy drive evolution equations?

Mark Peletier
TU Eindhoven

Many of the gradient-flow equations that we study and love are driven by functions that (some) people call ‘entropy’. In this short philosophical talk I want to address the question “Why?”

More concretely, I want to talk about questions such as “what does it mean when someone calls something an ’entropy’?”, and “why are our evolutions *driven* by such ‘entropies’?” This question is made all the more relevant by the appearance of many different functional forms for these entropies. In addition, the GENERIC structure of thermodynamically consistent evolution equations brings its own particular meanings of ‘energy’ and ‘entropy’, and also here the interpretation sometimes is murky.

This short talk will be organized around a core example that illustrates both how the word ‘entropy’ should be interpreted and how and why such entropies turn up as drivers of our variational evolutions.

Transport distances and gradient flows on metric graphs

Jan-Frederik Pietschmann
Universität Augsburg

We study the extension of transport distances to metric graphs, i.e. graphs in which to each edge, a one-dimensional interval is associated. Our starting point is the notion of dynamic transport distance a la Benamou and Brenier, for which we show that it can be generalized to this setting. We are specifically interested in the case where an independent dynamic on the vertices is allowed. We show the well-posedness of the respective distance in this case and also study the variational structure of gradient flows as well as certain asymptotic limits.

This is joint work with M.Burger, G. Heinze, I. Humpert and A. Schlichting

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Regularity by duality for minimising movements with nonlinear mobility

Lorenzo Portinale
Universität Bonn

In this talk, we will talk about conservation laws that can be written as gradient flows with respect to a Wasserstein distance with nonlinear mobility. In particular, we discuss some ideas on how to infer regularity estimates on time-discretisation schemes, via two important tools: (dynamical) duality and comparison principles.

Smoothing effect and particle approximation for a nonlocal conservation law

Emanuela Radici
Università degli Studi dell'Aquila

Conservation laws with nonlocality in the flux appear in many applied contexts such as traffic flow, sedimentation processes, supply chain modelling, and crowd movements. The one-dimensional prototype of nonlocal scalar conservation law extensively diffuse in the recent literature about these topics is the following

$$\partial_t \rho + \partial_x(\rho W[\rho]) = 0, \tag{CL}$$

where the nonlocal operator $W : \mathcal{P}(\mathbb{R}) \to L^\infty(\mathbb{R})$ is defined on the space of probability measures $\mathcal{P}(\mathbb{R})$ by $W[\rho] = v(V * \rho)$. We consider a standard set of assumptions on the potential V and on the velocity map v. More precisely, we require $V \geq 0$ to be compactly supported in $(-\infty, 0]$ and left-continuous in 0 with $V(0^-) > 0$, moreover V is $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and Lipschitz continuous in $(-\infty, 0]$. The velocity map $v : [0, \infty) \to \mathbb{R}$ is supposed to be Lipschitz continuous with $v' \leq b$ a.e. on $[0, \infty)$ for some $b > 0$. In this setting we can prove that the Cauchy problem for (CL) is well posed in the 2-Wasserstein space for measure initial data $\rho_0 \in \mathcal{P}(\mathbb{R})$. Moreover, all solutions satisfy an instantaneous smoothing effect from $\mathcal{P}(\mathbb{R})$ to $L^\infty(\mathbb{R})$. Finally, we provide a deterministic particle approximations of (CL), in the spirit of a Follow-the-Leader type ODE system, which catches at a discrete level the same smoothing effect.

This is a joint work with M. Di Francesco and S. Fagioli.

Log-sobolev inequalities using generalized JKO schemes

Filippo Santambrogio

Université Claude Bernard - Lyon 1

My talk will be focused on proving some functional inequalities (essentially, of log-Sobolev type) using the JKO scheme. It is well-known that the standard quadratic log-Sobolev inequality can be proven by differentiating both the entropy and the Fisher information along the Fokker-Planck equation. The same can be done on the JKO scheme, using some well-known computations based on a variant of the flow-interchange technique and on the five-gradients inequality. After presenting this, I will move to the generalized gradient flows using different transport costs and show a result (obtained in a recent preprint with Thibault Caillet) about the monotone behaviour of many Fisher informations along many doubly non-linear flows. I will then explain how to use these arguments, as we do in a work in progress with Thibault Caillet and Fanch Coudreuse, in order to obtain log-Sobolev inequalities for measures of the form e^{-V} where V is not necessarily uniformly convex. Most of the results that we obtain are well-known among specialists, but there is room for improvement about the values of the constants, and the technique is new. In particular, in some cases the use of a discrete (in time) JKO scheme instead of the continous flow is crucial (i.e., one should not send τ to zero).

Gradient Flow Solutions for Porous Medium Equations with Nonlocal Lévy-type pressure

Markus Schmidtchen

Technische Universität Dresden

We study a porous medium-type equation whose pressure is given by a nonlocal Lévy operator associated to a symmetric jump Lévy kernel. The class of nonlocal operators under consideration appears as a generalization of the classical fractional Laplace operator. For the class of Lévy-operators, we construct weak solutions using a variational minimizing movement scheme. The lack of interpolation techniques is ensued by technical challenges that render our setting more challenging than the one known for fractional operators.

Discrete-to-continuum limit for reaction-diffusion systems via variational convergence of gradient systems

Artur Stephan

Technische Universität Wien

My talk is about a convergence result for the spatial discretization of a reaction-diffusion system. The approximation is based on a homogeneous lattice, where in each node a reaction-ODE system describes the evolution of the concentrations, and the interaction between the different lattice nodes is given by additional exchange reactions.

Assuming detailed balance for the discrete system, the large coupled reaction system has a gradient structure, which will be the starting point for our analysis. Using the cosh-gradient structure for reactions, and sending the lattice length to zero, we show how the limit reaction-diffusion-PDE system can be derived. The proof is purely variational and relies on the energy-dissipation balance for gradient systems.

The talk is based on joint work with Georg Heinze and Alexander Mielke.

Keller-Segel type approximation for nonlocal Fokker-Planck equations in one-dimensional bounded domain

Yoshitaro Tanaka
Future University Hakodate

Motivated by various phenomena such as cell migration, cell adhesion and collective motion, many evolutional equations with convolution-type interactions are proposed. This type of the interaction is called the nonlocal interaction, and it is imposed as a velocity of the advection term in the models that target the above phenomena. In these evolution equations, by changing the shape of the integral kernel, it is possible to change the phenomenon being described and also to reproduce the various patterns as the solutions. In this talk, motivated by converting the nonlocality in the advection term by the spatially local effect, we approximate the nonlocal Fokker-Planck equation by the Keller-Segel system with multiple chemotactic factors in a singular limit analysis. In particular, by controlling the parameters in the Keller-Segel system, we show that any even integral kernels can be approximated by the linear sum of the fundamental solutions for an elliptic equation. From this result, we explain the mathematical relationship between the processes by the aggregation-diffusion with Haptotaxis and chemotaxis.

This talk is based on the result of the collaboration with Prof. Hideki Murakawa from Ryukoku University.

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