Quantum bosonic Fourier codes

how to design codes with nice logical gates?

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Basics of fault tolerance

- goal: compute on data protected by quantum error correction
- find “nice” implementations of logical gates
  - low-depth circuits, *e.g.* transversal gates
  - natural physical operations, *e.g.* linear optics (beasmplitters, phase-shifts)
  - “continuous”?

A natural problem

- the logical system:
  - a qudit $\mathbb{C}^d$
  - a logical group $G \subseteq SU(d)$
    e.g., single-qubit Clifford group
- the physical system
  - a physical space $\mathcal{H}$: Fock space, n-qudit space...
  - a nice physical representation $g \mapsto \rho(g)$
    e.g., Gaussian unitaries, transversal gates $\rho(g) = g^{\otimes n}$
- design an encoding $\mathcal{E} : \mathbb{C}^d \to \mathcal{H}$ where logical g is implemented with $\rho(g)$?

$$\boxed{\mathcal{E}(g|\psi\rangle) = \rho(g)\mathcal{E}(|\psi\rangle)}$$

studied by Jonathan Gross in spin encodings PRL 21

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Some limitations

- Eastin-Knill: cannot get a universal gate set with transversal implementation that corrects all relevant errors
- baby version:
  - cannot take G infinite and ρ continuous
  - otherwise, pick g, g' close
  - then $\rho(g)$ and $\rho(g')$ are also close (compared to experimental precision)

    $\implies$ $\rho(g)\mathcal{E}(|\psi\rangle)$ and $\rho(g')\mathcal{E}(|\psi\rangle)$ cannot both be prepared with arbitrary precision
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General recipe (Denys, AL, PRL 2024)

- group $G \subseteq U(d)$
- nice physical unitary representation ρ on $\mathcal{H}$
- goal: $\mathcal{E}(g|\psi\rangle) = \rho(g)\mathcal{E}(|\psi\rangle)$
- pick any logical state $|\Omega\rangle \in \mathbb{C}^d$ and any physical state $|\Phi\rangle \in \mathcal{H}$ (e.g. vacuum, coherent state)

Encoding map

$$\begin{aligned} \mathcal{E}: \quad \mathbb{C}^d &\rightarrow \mathcal{H}_P \\ |\psi\rangle &\mapsto \mathcal{E}(|\psi\rangle) \propto \textstyle\sum_{g \in G} \langle\Omega|g^\dagger|\psi\rangle\, \rho(g)|\Phi\rangle \end{aligned}$$

The case of bosonic codes

Pick

- $|\Phi\rangle = |\vec{\alpha}\rangle$ a (single-mode or multimode) coherent state
- $\rho(g)$: Gaussian passive unitary: $\rho(g)|\vec{\alpha}\rangle = |g\vec{\alpha}\rangle$

$$\mathcal{E}(|\psi\rangle) \propto \sum_{g \in G} \langle \Omega | g^{\dagger} | \psi \rangle \, |g\vec{\alpha}\rangle$$

is a superposition of coherent states.

$\Longrightarrow$ generalization of quantum spherical codes (Jain et al, Nat. Phys. 2024), but with nice gate sets.

one can recover the usual suspects:

- GKP: Weyl-Heisenberg group and displacements
- cat codes: cyclic group and phase shifts

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Elementary representation theory

Representation $\cong$ sum of irreducible representations

$$\rho(g) = U\left(\bigoplus_{\lambda \in \hat{G}} \lambda(g) \otimes \mathbb{1}_{M_\lambda}\right) U^\dagger$$

- U intertwiner
- λ: irreducible representation of G
- M_λ: multiplicity of λ in $\rho(g)$

One can rewrite the encoding as an isometry

$$\begin{aligned} \mathcal{E} \quad &: \mathbb{C}^d \to \mathcal{H} \\ &|\psi\rangle \mapsto U|\psi\rangle|\phi\rangle \end{aligned}$$

for some $|\phi\rangle \in \mathcal{M}$, multiplicity space associated to $\lambda \in \hat{G}$.

$\implies$ information encoded in an irreducible representation of G

Beyond bosonic codes

The construction is very general:

$$\mathcal{H}_L = \mathbb{C}^d, \qquad \mathcal{H}_P = (\mathbb{C}^{d'})^{\otimes n}$$

Natural choice for the physical representation $\rho(g)$:

- transversal gates $\rho(g) = g^{\otimes n}$

see works of Kubischta, Teixeira PRL 23, PRL 24

Code $[\![5,1]\!]$ with transversal $2T$

- $2T$ group: binary tetrahedral group, $2T = \langle Z, H\rangle = (\text{Paulis} + \text{Hadamard})$, $\quad |2T| = 24$
- choose the irrep $\rho_5(Z) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\quad \rho_5(H) = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$
- $\rho(g) = \rho_5(g)^{\otimes 5}$
- encoding: $\mathcal{E}(|\psi\rangle) = U|\psi\rangle|\phi\rangle$, for some $|\phi\rangle \in \mathcal{M}_5 \cong \mathbb{C}^6$
- can we get an encoding that corrects errors?
- asked chatGPT 5 to write a Python script that searches this space for encodings that satisfy the Knill-Laflamme conditions for single-qubit errors

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2O group: binary octahedral group (aka single-qubit Clifford group)

$$2\mathrm{O} = \langle \mathrm{S}, \mathrm{H} \rangle, \qquad |2\mathrm{O}| = 48$$

$$\rho_7^{\otimes 7} = \rho_6^{\oplus 7} \oplus \rho_7^{\oplus 15} \oplus \rho_8^{\oplus 21}$$

ρ_6, ρ_7: dimension 2, ρ_8: dimension 8; encode information in ρ_7

$$\rho(\mathrm{g}) = \rho_7(\mathrm{g})^{\dagger \otimes 7} \implies \text{standard Steane code} \quad [\![7,1,3]\!]$$

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again, need to optimize a state in the multiplicity space $\mathbb{C}^{15}$ to recover codes with large distance

not clear whether there exists a systematic way to guess which state $|\Phi\rangle$ yields a code with good error correction performance. Found some non stabilizer one with ChatGPT, but didn't investigate further.

attempts at finding codes with good distance: Kubischta, Teixeira PRL 24, Aydin, Albert, Barg 25

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Fourier codes

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An interesting case: ρ acts like the left-regular representation

- physical states indexed by group elements $|\psi_g\rangle$
- left regular representation: $\forall g, h \in G$,

$$\rho(g)|\psi_h\rangle = |\psi_{gh}\rangle$$

$\implies$ intertwiner $U = F_G^\dagger$, the inverse Fourier transform

$$F_G := \sum_{g\in G} \sum_{\lambda\in\hat{G}} \sqrt{\frac{d_\lambda}{|G|}} \sum_{\ell,m=1}^{d_\rho} \langle \ell|\lambda(g)|m\rangle \, |\lambda, \ell, m\rangle\langle g|$$

An example: Bosonic Fourier codes

- bosonic encoding and ρ = passive Gaussian operations

$$\rho(\mathrm{g})|\vec{\alpha}\rangle = |\mathrm{g}\vec{\alpha}\rangle$$

- define an orthonormal basis indexed by group elements:

$$|\psi_{\mathrm{g}}\rangle := \sum_{\mathrm{h}\in\mathrm{G}} [\Gamma^{-1/2}]_{\mathrm{h},\mathrm{g}} |\mathrm{h}\boldsymbol{\alpha}\rangle \qquad \Gamma_{\mathrm{h},\mathrm{g}} := \langle \mathrm{h}\alpha | \mathrm{g}\alpha\rangle$$

- encoding:

$$\mathcal{E}(|\ell, \mathrm{m}\rangle) = \mathrm{F}_{\mathrm{G}}^{\dagger} |\lambda, \ell, \mathrm{m}\rangle = \sum_{\mathrm{g}\in\mathrm{G}} [\Gamma^{-1/2}\mathrm{F}_{\mathrm{G}}^{\dagger}]_{\mathrm{g},\lambda\ell\mathrm{m}} |\mathrm{g}\boldsymbol{\alpha}\rangle$$

- encodes a logical and a multiplicity/auxiliary qubit: $|\ell\rangle, |\mathrm{m}\rangle \in \mathbb{C}^{\mathrm{d}}$

2-mode Fourier cat code: $\mathrm{G} = \langle \mathrm{X}, \mathrm{Z} \rangle$ with Gaussian unitaries

- pick $|\boldsymbol{\alpha}\rangle = |\alpha, \mathrm{i}\alpha\rangle$ with $\alpha = \sqrt{\frac{\pi}{2}}$

$$|\widehat{0,0}\rangle = |1_\alpha\rangle|0_{\mathrm{i}\alpha}\rangle, \quad |\widehat{0,1}\rangle = |1_{\mathrm{i}\alpha}\rangle|0_\alpha\rangle, \quad |\widehat{1,0}\rangle = |0_{\mathrm{i}\alpha}\rangle|1_\alpha\rangle, \quad |\widehat{1,1}\rangle = |0_\alpha\rangle|1_{\mathrm{i}\alpha}\rangle$$

- state preparation is ok: product states of cat states $|\mathrm{m}_\alpha\rangle \propto |\alpha\rangle + (-1)^{\mathrm{m}}|-\alpha\rangle$
- Pauli logical operations: SWAP for X, $(-1)^{\mathrm{n}_2}$ for Z
- S and CZ gates with Kerr interactions (standard for bosonic codes)
- $\mathrm{e}^{\mathrm{i}\theta \mathrm{Z}_\mathrm{L} \mathrm{Z}_\mathrm{M}}$: quantum Zeno effect
- more interesting: Hadamard $\implies$ universal gate set

Hadamard gate on the Fourier cat code

- applying $\rho(\mathrm{H})$ doesn't give a logical operation since $\mathrm{H} \notin \mathrm{G}$
- but it's close: $\mathrm{H_L H_M}$ + constellation rotation

  $\implies$ code deformation

- holds more generally for gates in $\mathrm{N(G)}$ (group normalizer)
- one can get $\mathrm{H_L}$ via $\mathrm{SHSHS} = e^{i\pi/4}\mathrm{H}$

$$\mathrm{H_L} = \mathrm{S_L}(\mathrm{H_L H_M})\mathrm{S_L}(\mathrm{H_L H_M})\mathrm{S_L}$$

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Bonus: implementation should be ok

Stabilizers, Lindblad operators, logical states of cat-type bosonic codes

	4-legged cat	**2-repetition cat**	**pair-cat**	**2-mode Fourier**
Stabilizers	$\hat{n} \mod 2$	$\hat{n}_1 - \hat{n}_2 \mod 2$	$\hat{n}_1 - \hat{n}_2$	$\hat{n}_1 - \hat{n}_2 \mod 2$
Lindblad operators	$\hat{a}^4 - \alpha^4$	$\hat{a}_1^2 - \alpha^2$ $\hat{a}_2^2 - \alpha^2$	$\hat{a}_1^2 \hat{a}_2^2 - \alpha^4$	$\hat{a}_1^4 - \alpha^4$ $\hat{a}_1^2 \hat{a}_2^2 + \alpha^4$
$\vert 0\rangle_L$	$\vert 0_\alpha\rangle$	$\vert 0_\alpha\rangle \vert 0_\alpha\rangle$	$\int_0^\pi \vert 0_{\alpha e^{i\theta}}\rangle \vert 0_{\alpha e^{-i\theta}}\rangle \, d\theta$	$\vert 1_\alpha\rangle \vert 0_{i\alpha}\rangle, \vert 1_{i\alpha}\rangle \vert 0_\alpha\rangle$
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Summary

- general formalism to design “codes” with specific physical representation of logical gates
- recovers the standard bosonic codes (GKP, cat codes)
- Fourier codes: reasonably nice universal gate set with the help of the multiplicity qubit

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- general approach to get a universal gate set (still *ad hoc* at the moment)?

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