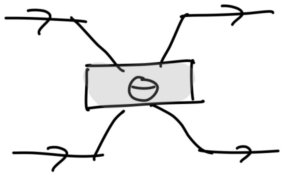
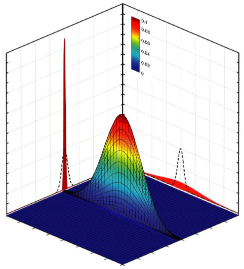
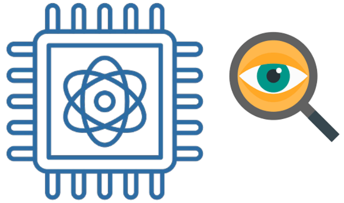
  

Robust Tests of Gaussianity

Michael Walter

Munich, Oct 2025

Joint works with Filippo Girardi, David Gross, Francesco Mele, Sepehr Nezami, Freek Witteveen, Lennart Bittel, Salvatore Oliviero

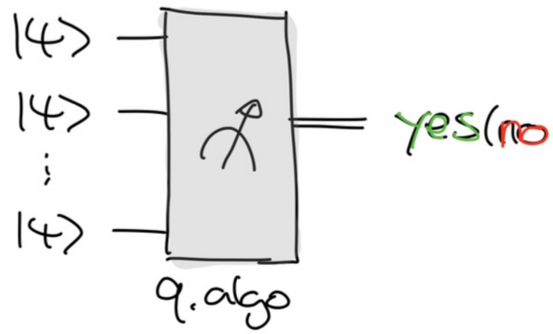
RUHR UNIVERSITÄT BOCHUM RUB → LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

erc European Research Council Established by the European Commission

Property testing

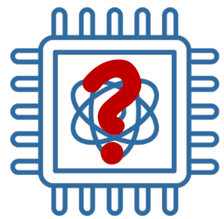
For a set X of quantum states, want algo that takes copies of *unknown state* ρ as input and decides between:

> **Yes**, ρ is in X
>
> **No**, ρ is ϵ-far from X



Question: Which properties of quantum states can be tested **efficiently**?

small # of samples (copies), simple circuits, …



Why care? Conceptually interesting, but also tells us which many-body properties that can be *practically* verified…

Many variations, e.g. tolerant testing, streaming model, unitary queries, …

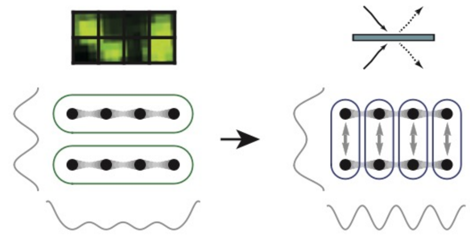
Property testing in the real world

Article | Published: 02 December 2015

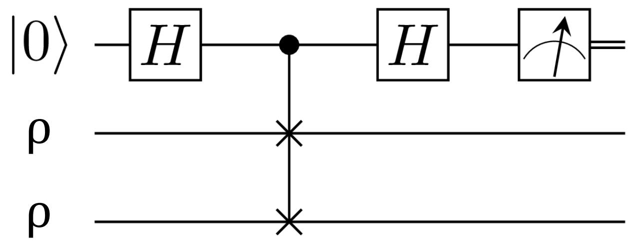
Measuring entanglement entropy in a quantum many-body system

Rajibul Islam, Ruichao Ma, Philipp M. Preiss, M. Eric Tai, Alexander Lukin, Matthew Rispoli & Markus Greiner

Nature **528**, 77–83 (2015) | Cite this article



Swap Test: uses 2 copies, acceptance probability related to **purity** tr ρ^2



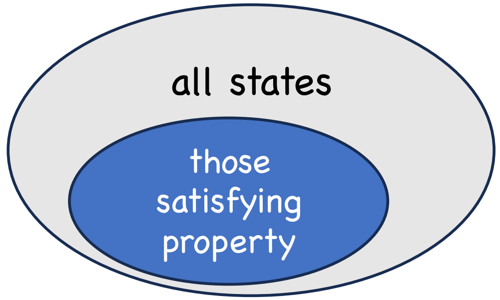
➔ "O(1) copies suffice to test if state is pure (or far from it)"

More precisely: O(1/eps) copies suffice to test if pure or eps-far from pure, with constant probability of error

➔ useful not just in practice, but also in theory!

Buhrman et al, Harrow-Montanaro, Watrous, …

Examples and surprises

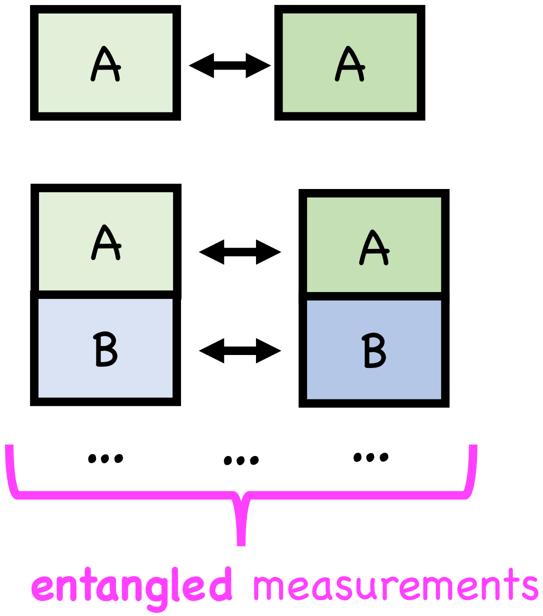


Sometimes few samples suffice, and sometimes not:

Purity: $O(1)$ copies ☺
Buhrman et al

Product: $O(1)$ copies ☺
Brandao-Harrow

Mixedness: $\Theta(2^n)$ copies ☹
Childs et al



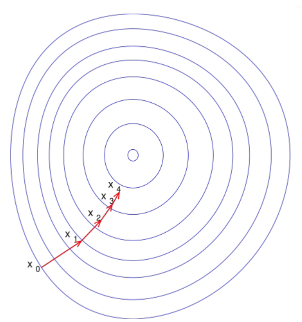
What if we restrict to **single-copy measurements**? In this case there can be an exponential disadvantage! ☹

Qubits vs bosons

Quantum computing is best developed in **finite dimension** $(\mathbb{C}^2)^{\otimes n}$
➔ quantum circuits, universality, complexity theory, ...

For bosonic (a.k.a. continuous-variable) quantum systems
even the right notion of complexity is not so clear (to me). $L^2(\mathbb{R}^n)$

→ talks by Simon, Ulysse, ..., recent work by Robert et al



In contrast, classical researchers routinely design algorithms that work with real numbers – think gradient descent!

Property testing and learning tasks can provide useful proving ground:
sample complexity already interesting, algos often turn out "practical"...

Gaussian states and unitaries

$L^2(R^n)$

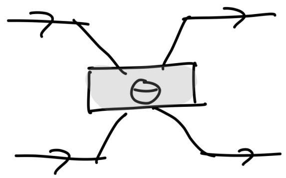
$X_1,\ldots,X_n,$
$P_1,\ldots,P_n$
$R_1,\ldots,R_{2n}$

A pure state is Gaussian if given by (complex) multivariate Gaussian wavefunction.

➔ Fully described by 2n-dimensional mean and covariance:

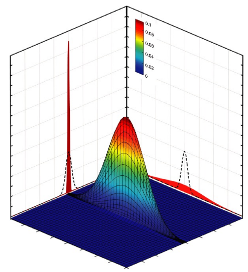
$\mu_j = \mathrm{tr}\ \rho R_j$ $\Sigma_{ij} = \mathrm{tr}\ \rho\{R_j - \mu_j, R_k - \mu_k\}$

➔ Generated by Gaussian unitaries a.k.a. linear quantum optics *(beam splitters, squeezing, …):*



$\mu \to S\mu + d$
$\Sigma \to S\Sigma S^T$

where S = *symplectic* matrix



➔ Phase

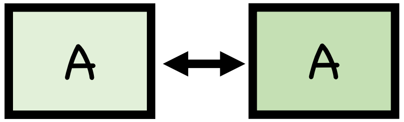
Question: Can we efficiently test if a given bosonic quantum state is Gaussian, or far from it?

Classically simulable. Very similar to Clifford unitaries & stabilizer states.

Warmup: Testing by symmetry

Recall **purity**:

ρ is pure $\Leftrightarrow$ $F\,\rho^{\otimes 2} = \rho^{\otimes 2}$ swap-invariant

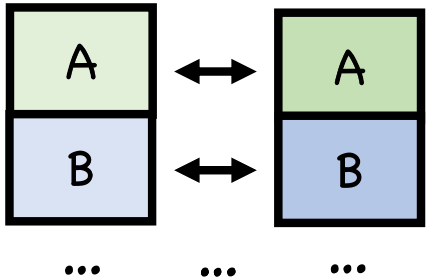


$U(d)$

S_2

Recall **productness**:

ρ is pure product state $\Leftrightarrow$ $F_A\,\rho^{\otimes 2} = F_B\,\rho^{\otimes 2} = \ldots = \rho^{\otimes 2}$ locally swap-invariant



$U(d_A) \times U(d_B) \times$

$S_2 \times S_2 \times$

In both cases:
- states form a single group orbit
- two copies have an enhanced symmetry ➔ natural test
- it is true, but not (fully) obvious that this test is robust

Smaller group $\Leftrightarrow$ subset of states $\Leftrightarrow$ larger symmetry.

Symmetry of Gaussians

Classical facts: If X is Gaussian with mean μ & covariance Σ...

linear transformations:

$\mu \to L\mu$
$\Sigma \to L\Sigma L^T$

t copies are again Gaussian

$\mu \to \mu \otimes 1_t$
$\Sigma \to \Sigma \otimes I_t$

t copies have enhanced symmetry
& this characterizes Gaussians!

permutations in S_t → orthogonal matrices in $O(t)$

stochastic if $\mu \neq 0$

In fact, a **45 degree rotation** is enough (if $\mu=0$).

$(X,Y) \to (X+Y,-X+Y)/\sqrt{2}$

Folklore: These are also quantum facts ☺

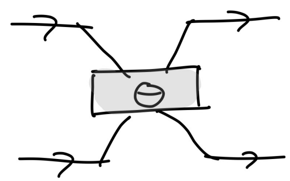
cf. Kac-Bernstein, Diaconis' Gaussian de Finetti, ...

Result: Gaussianity testing by symmetry

Quantum fact: A pure state ρ is Gaussian $\Leftrightarrow$ $\rho^{\otimes t}$ invariant under (stochastic) rotations in $O(t)$

$L^2(R^n)^{\otimes t}$
$= L^2(R^{n\times t})$

cf. Springer et al, Wolf et al (Gaussian extremality), König-Smith (entropy power), Leverrier (Gaussian q. de Finetti), Cuesta ("robust" fact), Bu-Li, Hahn-Takagi (test), ..., *hands-on calculation* ☺



We show that this gives rise to an efficient test:

Result: $O(\max(\varepsilon^{-2}, n^4E^4))$ copies suffice for Gaussianity, via rotation test that uses t=2 (3) copies at time.

E = "energy" per mode

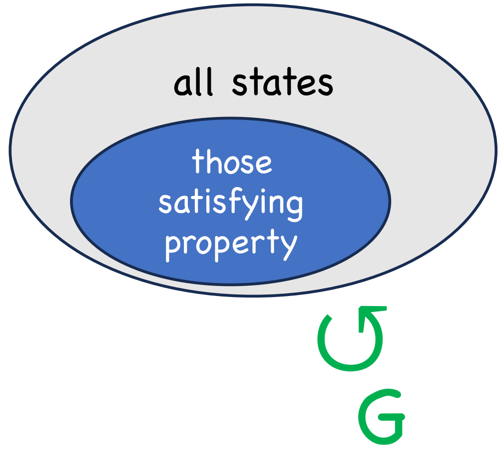
Intuition: $G = XP - PX$ generator of 2d rotations. WLOG Σ diagonal. Then:

$$\langle G^2\rangle = \langle X^2 + P^2\rangle^2 - 1$$

harmonic oscillator, gapped, ground state Gaussian

We also give a "tolerant" tester with similar guarantees.

Yoga of the commutant 🧘



General setup:

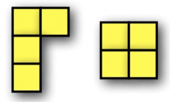
- A group G acts on the single-copy Hilbert space H
- Property is G-invariant (e.g., a single orbit)

Principle: Optimal t-copy test can always be taken in commutant of t^{th} tensor-power action.

$g^{\otimes t} \circlearrowleft H^{\otimes t}$

$[???, g^{\otimes t}] = 0$

Moreover, "generic" operator is natural candidate for test!



➔ purity and product testing: $U^{\otimes t}$ *vs* S_t — Schur-Weyl

➔ Gaussianity testing: $U_{Gaussian}^{\otimes t}$ *vs* stochastic $O(t)$ — Kashiwara-Vergne-Howe
also for fermions

In fact, same strategy applied to **stabilizer testing** motivated this work in the first place.

Gross-Nezami-W, Nebe-Scheeren, Bittel et al

➔ many applications in quantum TCS, many-body physics, ...

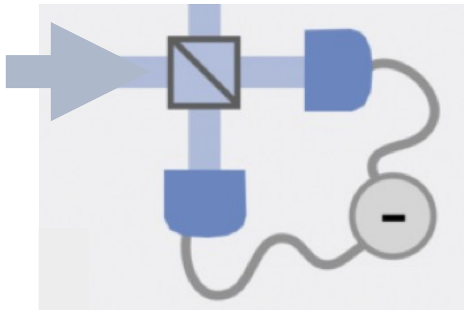
Result: Unentangled measurements

Recall that Gaussian states are described by covariance **Σ** (and mean).

Minimal uncertainty principle: For Gaussian states, the *symplectic* eigenvalues of **Σ** are = 1, and otherwise > 1.

$\Sigma \geq i\,\Omega$

Idea: Tomograph **Σ** using "homodyne" measurements, and test if symplectic eigenvalues ≈ 1.



Mele et al

Result: ε^{-8} poly(E,n) copies suffice to test Gaussianity *using single-copy measurements.*

Similarly, O(n) copies suffice for single-copy stabilizer testing. Hinsche-Helsen

Result: Lower bounds

We saw: Gaussianity can be tested efficiently, using poly(n,E) copies.

Question: Is Gaussianity testing even possible with # of copies that is *independent* of # of modes and energy?

Partial answer: Yes, if $\varepsilon \leq \varepsilon_0$ using the 45-degree rotation test.

There are also Gaussian mixed states. Can those be tested efficiently?

"No go" result: Even restricted to bounded energy states, exp(n) copies are required to test if a mixed state is Gaussian or 1/poly(n)-far from it.

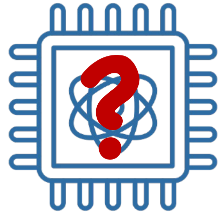
can this be constant?

Rough idea: Valiant-Valiant construct hard-to-distinguish classical distributions, from "any" starting distribution.

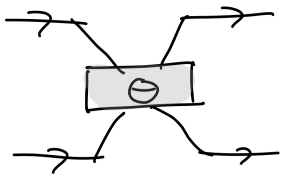
➔ Apply to squared amplitudes of thermal state

➔ Good *quantum* Gaussianity test would imply *classical* contradiction.

Summary



Property testing asks which many-body properties can be practically verified, and which *cannot*.



Here we focused on Gaussian states, which are of conceptual interest very widely used.

We found new mathematical tools and quantum protocols to robustly verify Gaussianity, and a “no go” for mixed states.

Symmetry and learning theory techniques that could be of independent interest. Many interesting open problems...

Thank you for your attention!