Exercises, lecture 1

① Let X be a Banach space with dual X^* and $X^{**} := (X^*)^*$

ⓐ For fixed $x \in X$ let $\delta_x : X^* \to \mathbb{R}$ with $\delta_x(x^*) := \langle x^*, x \rangle$. Show that $\delta_x \in X^{**}$.

ⓑ Show that $x_n^* \rightharpoonup x^* \Rightarrow x_n^* \xrightharpoonup{*} x^*$

ⓒ Show that $x_n^* \to x^*$ (in norm) $\Rightarrow x_n^* \rightharpoonup x^*$

ⓓ Prove that norm-continuity of a functional $F : X^* \to \mathbb{R}$ is weaker than weak continuity.

② Assume that X has a predual (and the norms coincide) and let $F(x) := \|x\|_X$.

ⓐ Show that F is weakly-* l.s.c.

ⓑ Show that there exists a minimiser of F (Using the direct method, else it would be trivial!)

③ Can you construct a convex discontinuous function $F : \mathbb{R} \to \mathbb{R}$...?

Exercises Lecture 2

① What are the weak and weak-$*$ topologies on $\mathbb{R}$?

② By Riesz' Representation Theorem for measures,

$$(C_c(\mathbb{R}))^* \simeq \{\mu : \text{regular signed measures on } \mathbb{R}\},$$

with

$$\|\mu\|_{C_c^*} = \sup_{\substack{\varphi \in C_c \\ \|\varphi\|_{L^\infty} \le 1}} \langle \varphi, \mu \rangle = |\mu|(\mathbb{R}) \text{ "total variation".}$$

Consider the sequence $(\delta_n)_{n\in\mathbb{N}} \subset (C_c(\mathbb{R}))^*$.

ⓐ Show that the sequence is weakly-$*$ compact.

ⓑ The sequence actually converges weakly-$*$ (not just up to a subsequence). Calculate the limit.

ⓒ By another Riesz Representation Theorem,

$$(C_b(\mathbb{R}))^* \simeq \{\mu : \text{regular, finitely additive set functions}\}$$

(i.e. "measures" that only finitely additive)

Show that the sequence $(\delta_n)_{n\in\mathbb{N}} \subset (C_b(\mathbb{R}))^*$ is weakly-$*$ compact.

③ Show that the Hahn-Banach seperation theorem implies that:

- $\operatorname{dist}(A, H) \ge \varepsilon$
- $\operatorname{dist}(B, H) \ge \varepsilon$.

④ ⓐ Let $F : X \to \mathbb{R} \cup \{\infty\}$ be convex. Show that its level sets $\{F \le C\}$ are convex

ⓑ Give an example of a non-convex function with convex level sets.

Exercises Lecture 3

1) Let $F: X \to \mathbb{R} \cup \{\infty\}$ be convex & l.s.c. (in the norm topology). Show that F is weakly l.s.c.

2) Show that F is convex $\Leftrightarrow$ epi(F) is convex.

3) Consider the functional $F(x) \equiv \infty$.

(a) Is F convex?

(b) Is F lsc?

(c) What is epi(F)?

Exercises 4

1. If $\liminf_{n\to\infty} a_n = a = \limsup_n a_n$ $\quad ((a_n)_{n\in\mathbb{N}} \in \mathbb{R})$ $\quad (a \in \mathbb{R})$

   then $\lim_{n\to\infty} a_n = a$.

2. (a) If $F: X \to \mathbb{R}$ is lower and upper semicontinuous then F is continuous (in some topology).

   (b) If $F: X \to \mathbb{R}$ is sequentially lsc & usc then F is sequentially continuous.

3. If $F: \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is convex, then it is weakly lsc on $\operatorname{int}(\operatorname{dom} F)$.

Exercises (5)

(1) $\lambda_B : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ "The Boltzmann function"

$$\lambda_B(x) := \begin{cases} x\log x - x + 1, & x > 0, \\ 1, & x = 0, \\ \infty, & x < 0. \end{cases}$$

(a) Is λ_B convex?

(b) What is $\partial\lambda_B(x)$ for $x \in \mathbb{R}$?

(2) For any convex $F : X \to \mathbb{R} \cup \{\infty\}$, show that $\partial F(x)$ is sequentially weakly-$*$ closed.

(3) Prove the monotonicity property of subdifferentials.

(4) Prove Fermat's Rule.

Exercises 6

1) Let $C \subset X$ be a set, and

$$F(x) = \delta_x(C) = \mathbb{1}_{\{x \in C\}}$$

ⓐ Calculate $\bar{F}$

ⓑ Calculate $\operatorname{co} F$

ⓒ Write F^* in terms of the support function

2) Let $X = \mathbb{R}$ and $F: X \to \mathbb{R}$ (not attaining ∞) continuous differentiable and convex.

Construct an explicit family $(F_{x_0})_{x_0 \in X}$ of continuous affine functions such that $F(x) = \sup_{x_0 \in X} F_{x_0}(x)$.

3) Let X be any Banach space and $F: X \to \mathbb{R}$ be lsc and convex. In Lecture 5 we proved that the subdifferential is (everywhere) convex, (norm-) bounded and nonempty. Use the subdifferential to construct a family $(F_i)_{i \in I}$ of continuous affine functions such that $F(x) = \sup_{i \in I} F_i(x)$.

(Hint: this requires a larger index set than in the previous exercise, or an explicit use of the axiom of choice!)

4a) $X = \mathbb{R}$, $b > 0$.

Calculate the convex dual of $F(x) = b(e^x - 1)$.

b) Calculate the convex dual of

$$F(x) = b\lambda_B\left(\frac{x}{b}\right) = \begin{cases} b, & x = 0 \\ x \log\frac{x}{b} - x + b, & x > 0 \\ \infty, & x < 0. \end{cases}$$

Exercises 7

① Prove the proposition "properties of convex duals"

② Assume that a convex $F: X \to \mathbb{R} \cup \{\infty\}$ and its convex dual $F^*: X^* \to \mathbb{R} \cup \{\infty\}$ are both Gâteaux differentiable. ~~and recall that $DF(x)$, $DF^*(x^*)$~~. What is the relation between DF and DF^*?

Exercises ρ

1a) Let $F : H \to \mathbb{R} \cup \{\infty\}$.

~~Calc~~ What is $(F_\varepsilon)_\varepsilon$ (The Moreau-Yosida regularisation applied to the Moreau-Yosida regularisation) ?

2a) Calculate F_ε^*

2b) Now assume F is proper, convex & lsc.
Calculate F_ε^{**}

Exercises 9

① & ② Prove the two corollaries of Jensen's inequality from Lecture 9. part B.

③ Show that (for $x \in L^2((0,1)^d)$):

$$\int_{(0,1)^d} \lambda_B(x(q))\,dq = \int_{(0,1)^d} \left(x(q)\log x(q) - x(q) + 1\right) dq \leq \|x\|_{L^1} \log \frac{\|x\|_{L^2}^2}{\|x\|_{L^1}} - \|x\|_{L^1} + 1$$

Exercise 10

1. Assume $U \subset \mathbb{R}^n$ is bounded, connected and has a smooth boundary. Recall Poincaré's inequality:

$$\boxed{\begin{array}{l}\text{For any } 1 \leq p \leq \infty \text{ there is a constant } C \text{ s.t. for all } x \in W^{1,p}(U):\\ \|x - \bar{x}\|_{L^p(U)} \leq \|\nabla x\|_{L^p(U)}, \quad \bar{x} := \frac{1}{|U|}\int_U x(q)\,dq.\end{array}}$$

Let $L(a,b) := \sqrt{|a|}\, e^{|a|}(\sin b + 1)$, and
$F(x) := \int_U L(\nabla x(q), x(q))\,dq$.

We shall show that the constrained minimisation problem $\inf_{\substack{x \in W^{1,1\frac{1}{2}}(U):\\ \bar{x} = 1}} F(x)$ admits a minimiser.

ⓐ First show that $G(x) := \begin{cases} F(x), & \bar{x} = 1, \\ \infty, & \bar{x} \neq 1 \end{cases}$ has weak sequentially relative compact level sets. (Hint: Banach-Alaoglu & Poincaré)

ⓑ Show that F is weakly (sequentially) lsc (in $W^{1,1\frac{1}{2}}(U)$).

ⓒ Show that $x \mapsto \bar{x}$ is (sequentially) weakly continuous, and deduce that $\{\bar{x} = 1\}$ is weakly (sequentially) closed.

ⓓ Show that G is (sequentially) weakly lsc.

ⓔ Use the direct method to show that the constrained minimisation problem has a solution.

Exercises 11

① Recall that $\lambda_B(z) = z\log z - z + 1$

and $z \mapsto \lambda_B(|z|+1)$ is an N-function.

ⓐ Exploit the convexity of λ_B to show that

$$\lambda_B(z) \geq 2\lambda_B\left(\tfrac{z}{2}+1\right) - \lambda_B\left(\tfrac{1}{2}\right).$$

ⓑ Let $F(x) := \int_{(0,1)^d} \lambda_B(x(q))\,dq = S(x \mid \mathcal{L}|_{(0,1)^d})$, for

$x \in L^1((0,1)^d)$, $x \geq 0$.

Show that $\int_{(0,1)^d} \lambda_B\left(\frac{|x(q)|}{2} + 1\right)dq$ is uniformly bounded

on level sets $\{F \leq C\}$.

ⓒ Show that $\{F \leq C\}$ is also uniformly $L^1((0,1)^d)$-bounded.

ⓓ Deduce that F has weakly compact level sets in $L^1((0,1)^d)$.

② Let φ be an N-function. Prove that φ^* is an N-function.

③ Show that any λ-convex functional can be written as ~~the~~ a supremum over quadratic functionals of the form

$$x \mapsto a\|x\|^2 + \langle x^*, x\rangle + b, \quad x^* \in X^*,\ a, b \in \mathbb{R}.$$

Exercises $\boxed{12}$

(1) We shall prove that $\|\cdot\|_\varphi$ is a norm (Assuming φ is an N-function).

(a) Prove that $\|\alpha x\|_\varphi = |\alpha| \, \|x\|_\varphi$.

(b) Prove the triangle inequality:
$$\|x+y\|_\varphi \leq \|x\|_\varphi + \|y\|_\varphi .$$

(c) Show that there exists a $c>0$ so that $\varphi(c) \leq 1$.

(d) Use this constant to show that $\|x\|_\varphi = 0 \Rightarrow x=0$ (μ-a.e.).

(2) Let
$$\Psi_1(z) := z \log 2z - z,$$
$$\Psi_2(z) := -z \log(-3z) - z,$$
$$\Psi(z) := (\Psi_1 \square \Psi_2)(z), \text{ and } F(x) := \int \Psi(|x|) d\mu .$$

We work with the N-function $\varphi(z) := \cosh^*(z) + 1$.
Calculating explicit expressions for Ψ and φ is a pain!
Instead, it's much easier to work with their convex duals!
We shall prove that F has N_φ-bounded level sets $\{F \leq C\}$.

(a) Calculate Ψ_1^*, Ψ_2^*, Ψ^*, and φ^*.

(b) Show that $\Psi^* \leq \varphi^* + 1$.

(c) Deduce that $\Psi \geq \ldots$

(d) Use this inequality (from (c)) to show that $\int \varphi(|x|) d\mu$ is uniformly bounded on level sets $\{F \leq C\}$.

(e) Deduce that $\|x\|_\varphi$ is uniformly bounded on level sets. (Krasnoselskii–Rutickii)

(f) Exploit the equivalence of norms to show that $N_\varphi(x)$ is uniformly bounded on level sets.

(3) (a) Use the unit ball property of the Luxemburg norm to deduce that
$$\|x\|_\varphi = \sup \left\{ \int |xy| d\mu : y \in L^{\varphi^*}_\mu(\Omega), \; N_{\varphi^*}(y) \leq 1 \right\}$$

(b) Deduce the Hölder-type estimate:
$$\int |xy| d\mu \leq \|x\|_\varphi N_{\varphi^*}(y) \wedge N_\varphi(x) \|y\|_\varphi .$$

Exercises $\boxed{13}$

① Consider the same setting as exercise 11.1; i.e.

$$\lambda_B(z) = \begin{cases} z\log z - z + 1, & z > 0, \\ 1, & z = 0, \\ \infty, & z < 0, \end{cases}$$ and

$$F(x) = \int_{(0,1)^d} \lambda_B(x)\, dx.$$

Let $\varphi(z) := \lambda_B(|z|+1)$ (this is also a typical N-function)

ⓐ prove that $F(x) < \infty \Rightarrow x \in L^1_\mu((0,1)^d)$.

ⓑ prove that $F(x) < \infty \Rightarrow x \in L^\varphi_\mu((0,1)^d)$.

ⓒ derive that F has N_φ-uniformly bounded level sets.

ⓓ deduce that F has L^1_μ-weakly compact sets.

ⓔ deduce that F has L^φ_μ-weakly-$*$ compact level sets.

ⓕ what could be a general strategy to prove L^1_μ-weak or L^φ_μ-weak-$*$ lower semicontinuity?