Exercises, lecture 1

1. Let $X$ be a Banach space with dual $X^*$ and $X^{**} := (X^*)^*$.
   a. For fixed $x \in X$ let $S_x : X^* \to \mathbb{R}$ with $S_x(x^*) := \langle x^*, x \rangle$.
      Show that $S_x \in X^{**}$.
   b. Show that $x^*_n \to x^*$ implies $x^*_n \to x^*$.
   c. Show that $x^*_n \to x^*$ (in norm) implies $x^*_n \to x^*$.
   d. Prove that norm-continuity of a functional $F : X^* \to \mathbb{R}$ is weaker than weak continuity.

2. Assume that $X$ has a predual (and the norms coincide) and let $F(x) := \|x\|_X$.
   a. Show that $F$ is weakly-* l.s.c.
   b. Show that there exists a minimizer of $F$ (Using the direct method, else it would be trivial!)

3. Can you construct a convex discontinuous function $F : \mathbb{R} \to \mathbb{R}$?
Exercises Lecture 2

1. What are the weak and weak-* topologies on IR?

2. By Riesz' Representation Theorem for measures,
\[(C_c(\mathbb{R}))^* = \{\mu: \text{regular signed measures on } \mathbb{R}\},\]
with
\[\|\mu\|_{C_c} = \sup_{\varphi \in C_c} \langle \varphi, \mu \rangle = \|\mu\|_1(\mathbb{R}) \text{ "total variation".}\]
Consider the sequence \((S_n)_{n \in \mathbb{N}} \subset (C_c(\mathbb{R}))^*\).

a. Show that the sequence is weakly-* compact.

b. The sequence actually converges weakly-* (not just up to a subsequence). Calculate the limit.

c. By another Riesz Representation Theorem,
\[(C_b(\mathbb{R}))^* = \{\mu: \text{regular, finitely additive set functions}\},\]
(i.e. "measures" that only finitely additive)
Show that the sequence \((S_n)_{n \in \mathbb{N}} \subset (C_b(\mathbb{R}))^*\) is weakly-* compact.

3. Show that the Hahn-Banach separation theorem implies that:
- \(\text{dist}(A, H) \geq \varepsilon\)
- \(\text{dist}(B, H) \geq \varepsilon\).

4. Let \(F: X \rightarrow [\mathbb{R} \cup \{\infty\}]\) be convex. Show that its level sets \(\{F \leq c\}\) are convex.

b. Give an example of a non-convex function with convex level sets.
1) Let $F: X \to \mathbb{R} \cup \{\infty\}$ be convex and l.s.c. (in the norm topology). Show that $F$ is weakly l.s.c.

2) Show that $F$ is convex $\iff$ epi$(F)$ is convex.

3) Consider the functional $F(x) = \infty$.
   a) Is $F$ convex?
   b) Is $F$ l.s.c.?
   c) What is epi$(F)$?
1. If \( \liminf_{n \to \infty} a_n = a = \limsup_{n \to \infty} a_n \) \( (a_n)_{n \in \mathbb{N}} \in \mathbb{R} \) \( a \in \mathbb{R} \)
then \( \lim_{n \to \infty} a_n = a \).

2. If \( F : X \to \mathbb{R} \) is lower and upper semicontinuous
then \( F \) is continuous \( \) (in some topology).

3. If \( F : X \to \mathbb{R} \) is sequentially lsc and usc
then \( F \) is sequentially continuous.

4. If \( F : \mathbb{R}^d \to \mathbb{R} \cup \{0\} \) is convex, then it is weakly
lsc on \( \text{int} \)(dom \( F \)).
Exercises 5

1. \( \lambda_B : \mathbb{R} \to \mathbb{R} \cup \{ \infty \} \) "The Boltzmann function"
\[ \lambda_B(x) = \begin{cases} x \log x - x + 1, & x > 0, \\ 1, & x = 0, \\ \infty, & x < 0. \end{cases} \]

2. Is \( \lambda_B \) convex?

3. What is \( \partial \lambda_B(x) \) for \( x \in \mathbb{R} \)?

4. For any convex \( F : X \to \mathbb{R} \cup \{ \infty \} \), show that \( \partial F(x) \) is sequentially weakly-* closed.

5. Prove the monotonicity property of subdifferentials.

Exercises 6

1) Let $C \subseteq X$ be a set, and

$$F(x) = S_x(C) = \mathbb{1}_{\{x \in C\}}$$

a) Calculate $F$

b) Calculate $\text{co}F$

c) Write $F^*$ in terms of the support function

2) Let $X = \mathbb{R}$ and $F : X \to \mathbb{R}$ (not attaining $\infty$) continuous, differentiable and convex. Construct an explicit family $(F_{x_0})_{x_0 \in X}$ of continuous affine functions such that $F(x) = \sup_{x_0 \in X} F_{x_0}(x)$.

3) Let $X$ be any Banach space and $F : X \to \mathbb{R}$ be lsc and convex. In lecture 5 we proved that the subdifferential is (everywhere) convex, (norm-)bounded and nonempty. Use the subdifferential to construct a family $(F_i)_{i \in I}$ of continuous affine functions such that $F(x) = \sup_{i \in I} F_i(x)$. (Hint: this requires a larger index set than in the previous exercise, or an explicit use of the axiom of choice!)

4a) $X = \mathbb{R}$, $b > 0$.

Calculate the convex dual of $F(x) = b(e^x - 1)$.

b) Calculate the convex dual of

$$F(x) = \begin{cases} b \lambda_b \left( \frac{x}{b} \right) & x > 0 \\ -x + b & x < 0 \end{cases}$$
1. Prove the proposition "properties of convex duals".

2. Assume that a convex $F : X \to \mathbb{R} \cup \{\infty\}$ and its convex dual $F^* : X^* \to \mathbb{R} \cup \{\infty\}$ are both Gâteaux differentiable, and recall that $DF(x), DF^*(x)$. What is the relation between $DF$ and $DF^*$?
1. Let \( F : H \to \mathbb{R}_+ \).

   a. What is \( (F_\varepsilon)_\varepsilon \) (The Moreau-Yosida regularization applied to the Moreau-Yosida regularization)?

2a. Calculate \( F_\varepsilon^* \)

2b. Now assume \( F \) is proper, convex & l.s.c.

   Calculate \( F_{\varepsilon}^{**} \)
1. & 2. Prove the two corollaries of Jensen's inequality from Lecture g, part B.

3. Show that (for $x \in L^2((0,1)^d)$):

$$\int_{(0,1)^d} x_1(x_2) dx = \int_{(0,1)^d} (x_2 \log x_2 - x_2 + 1) dx \leq \|x\|_L^2 \log \frac{\|x\|_L^2}{\|x\|_L^2} - \|x\|_L^2 + 1$$
Exercise 10

1. Assume \( U \subset \mathbb{R}^n \) is bounded, connected and has a smooth boundary. Recall Poincaré's inequality:

\[
\| x - \bar{x} \|_{L^p(U)} \leq \| \nabla x \|_{L^p(U)}, \quad \bar{x} := \frac{1}{|U|} \int_U x(u) \, du.
\]

Let \( L(a, b) := \sqrt{\lambda a e^{\lambda a} (\sin b + 1)} \), and
\[
F(x) := \int_U L(\nabla x(u), x(u)) \, du.
\]

We shall show that the constrained minimisation problem

\[
\inf_{x \in W^{1,1/2}(U), \bar{x} = 1} F(x)
\]

admits a minimiser.

(a) First show that \( G(x) := \begin{cases} F(x), & \bar{x} = 1, \\ \infty, & \bar{x} \neq 1 \end{cases} \) has weak sequential relative compact level sets. (Hint: Banach-Alaoglu & Poincaré)

(b) Show that \( F \) is weakly (sequentially) lsc (in \( W^{1,1/2}(U) \)).

(c) Show that \( x \mapsto \bar{x} \) is (sequentially) weakly continuous, and deduce that \( \{ \bar{x} = 1 \} \) is weakly (sequentially) closed.

(d) Show that \( G \) is (sequentially) weakly lsc.

(e) Use the direct method to show that the constrained minimisation problem has a solution.
Exercises 11

1. Recall that \( \lambda_0(z) = z \log z - z + 1 \) and \( z \mapsto \lambda_0(1z^1 + 1) \) is an N-function.

2. Exploit the convexity of \( \lambda_0 \) to show that
   \[ \lambda_0(z) \geq 2 \lambda_0(\frac{z}{2} + 1) - \lambda_0\left(\frac{z}{2}\right). \]

3. Let \( F(x) = \int \lambda_0(x(a)) \, dq = S(x \cdot \mathbf{1}((0,1)^d)) \), for \( x \in L^1((0,1)^d) \), \( x \geq 0 \).
   Show that \( \int \lambda_0(\frac{|x(a)|}{2} + 1) \, dq \) is uniformly bounded on level sets \( \{ F \leq C \} \).

4. Show that \( \{ F \leq C \} \) is also uniformly \( L^1((0,1)^d) \)-bounded.

5. Deduce that \( F \) has weakly compact level sets in \( L^2((0,1)^d) \).

2. Let \( \varphi \) be an N-function. Prove that \( \varphi^* \) is an N-function.

3. Show that any \( \lambda \)-convex functional can be written as the supremum over quadratic functionals of the form
   \[ x \mapsto a \| x \|^2 + \langle x^*, x \rangle + b, \quad x^* \in X^*, \quad a, b \in \mathbb{R}. \]
Exercises 12

1. We shall prove that \( \| \cdot \|_p \) is a norm (Assuming \( \varphi \) is an \( N \)-function).
   a) Prove that \( \| a \cdot x \|_p = |a| \| x \|_p \).
   b) Prove the triangle inequality:
      \( \| x + y \|_p \leq \| x \|_p + \| y \|_p \).
   c) Show that there exists a \( c > 0 \) so that \( \varphi(c) \leq 1 \).
   d) Use this constant to show that \( \| x \|_p = 0 \Rightarrow x = 0 \) (\( \mu \)-a.e.).

2. Let \( \psi_1(z) := z \log z - z \),
   \( \psi_2(z) := -z \log(32z) - z \),
   \( \psi(z) := (\psi_1 \circ \psi_2)(z) \), and \( F(x) := \int \psi(1 \times 1) \, dy \).
   We work with the \( N \)-function \( \varphi(z) := \cosh(1)(z) + 1 \).
   Calculating explicit expressions for \( \psi \) and \( \varphi \) is a pain!
   Instead, it's much easier to work with their convex duals!
   We shall prove that \( F \) has \( N_\varphi \)-bounded level sets \( \{ F \leq C \} \).
   a) Calculate \( \psi^* \), \( \psi_2^* \), \( \varphi^* \), and \( \varphi^* \).
   b) Show that \( \psi^* \leq \varphi^* + 1 \).
   c) Deduce that \( \psi \geq \ldots \).
   d) Use this inequality (from c) to show that \( \int \psi(1 \times 1) \, dy \) is uniformly bounded on level sets \( \{ F \leq C \} \).
   e) Deduce that \( \| x \|_p \) is uniformly bounded on level sets.
      (Krasnoselskii-Rutickii)
   f) Exploit the equivalence of norms to show that \( N_\varphi(x) \) is uniformly bounded on level sets.

3. Use the unit ball property of the Luxemburg norm to deduce that
   \( \| x \|_p = \sup \{ \int x \cdot y \, dy : y \in L^p(\mathbb{R}), \ N_\varphi(y) \leq 1 \} \).
   b) Deduce the Hölder-type estimate:
      \( \int x \cdot y \, dy \leq \| x \|_p \varphi^*(y) \wedge N_\varphi(x)\| y \|_p \).
1. Consider the same setting as exercise 11.1, i.e.
\[ \lambda_\beta(z) = \begin{cases} \frac{z \log z - z + 1}{z}, & \text{if } z > 0, \\ 0, & \text{if } z = 0, \\ \frac{z - 1}{z}, & \text{if } z < 0, \end{cases} \]
and
\[ F(x) = \int_{(0,1)^d} \lambda_\beta(x) \, dx. \]
Let \( \varphi(z) := \lambda_\beta(|z| + 1) \) (this is also a typical \( N \)-function).

(a) prove that \( F(x) < \infty \Rightarrow x \in \ell_1^1((0,1)^d) \).
(b) prove that \( F(x) < \infty \Rightarrow x \in \ell^0_\infty((0,1)^d) \).
(c) derive a that \( F \) has \( N_\infty \)-uniformly bounded level sets.
(d) deduce that \( F \) has \( L_1^1 \)-weakly compact sets.
(e) deduce that \( F \) has \( L^0_\infty \)-weakly-* compact level sets.
(f) what could be a general strategy to prove \( L_1^1 \)-weak or \( L^0_\infty \)-weak-* lower semicontinuity?