1. Let X_1 be a random variable with finite moments $\phi(t) := \mathbb{E}e^{tX_1} < \infty$, with its Cramér transform $\hat{X}_1$ (as defined in the lecture video L2a). Calculate $\mathbb{E}\hat{X}_1$ and $\hat{\phi}(t) := \mathbb{E}e^{t\hat{X}_1}$.

2. Let X_1 be a random variable with finite moments $\phi(t) := \mathbb{E}e^{tX_1} < \infty$. Show that $\log\phi$ is convex. *Hint: for a convex combination $(1-\lambda)t_0 + \lambda t_1$, apply Hölder's inequality with the two functions $e^{(1-\lambda)t_0 x}$ and $e^{\lambda t_1 x}$.*

Let $\mathcal{X}$ be a topological space equipped with its Borel σ-algebra.

Definition 0.1. *A sequence of random variables X_n (or its laws $\mu_n \in \mathbb{P}(\mathcal{X})$) is* exponentially tight *if for any $\eta < \infty$ there exists a compact*[1] *set $K_\eta \subset$ so that*

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mu_n(K_\eta^{\mathrm{c}}) < -\eta.$$

Definition 0.2. *A functional $I : \mathcal{X} \to [0,\infty]$ is called* lower semicontinuous *if the (sub)level sets $\{\mathcal{I} \le C\} := \{x \in \mathcal{X} : \mathcal{I}(x) \le C\}$ are closed.*
A functional $I : \mathcal{X} \to [0,\infty]$ is called good[2] *if the level sets $\{I \le C\}$ are compact.*

3. Let an exponentially tight sequence μ_n satisfy a large-deviation principle (actually we only need the lower bound!) with rate functional $\mathcal{I} : \mathcal{X} \to [0,\infty]$.
   (a) For any $\eta < \infty$, derive that $\inf_{K_\eta^{\mathrm{c}}} \mathcal{I} > \eta$,
   (b) Argue that the level set $\{\mathcal{I} \le \eta\}$ is contained in K_η,
   (c) Conclude that $\mathcal{I}$ is good.

Remark 0.3. *Actually, for Polish (=separable metric) spaces, exponential tightness and goodness of the rate functional are equivalent!*

[1]For the Borel σ-algebra, compact sets are automatically measurable.
[2]The term 'goodness' is used almost exclusively in large-deviations theory; in analysis it is sometimes called coercivity.