1. Prove the following *Laplace principle*

$$\lim_{n\to\infty} \tfrac{1}{n} \log\left(e^{-an} + e^{-bn}\right) = -\min\{a,b\} \qquad \text{for any } a,b \in [0,\infty). \tag{1}$$

(It's a very nice simple trick, but don't spend too much time on it if you don't see it ;-))

Let $\mathcal{X}$ be a topological space (with its Borel σ-algebra).

2. Let $(\mu_n)_n$ satisfy an LDP with rate functional $I : \mathcal{X} \to [0,\infty]$. Show that $\inf_{\mathcal{X}} I = 0$, and argue that this may not hold when $(\mu_n)_n$ satisfies a weak LDP only.

3. Let $(\mu_n)_n$ satisfy a *weak* LDP with rate functional $I : \mathcal{X} \to [0,\infty]$, and assume that $(\mu_n)_n$ is exponentially tight. We are going to show that $(\mu_n)_n$ satisfies an LDP.

   (a) Take any closed set $G \subset \mathcal{X}$ and let K_η be compact sets for which the sequence is exponentially tight. Write down the large-deviation upper bounds for the sets $G \cap K_\eta$ and K_η^{c}.

   (b) Derive upper bounds on $\mu_n(G\cap K_\eta)$ and $\mu_n(K_\eta)$ for n sufficiently large (this will involve an arbitrary $\epsilon > 0$).

   (c) Use this together with the Laplace principle (1) to show that

   $$\limsup_{n\to\infty} \tfrac{1}{n} \log \mu_n(G) \le -\min\{\inf_{G\cap K_\eta} I, \eta\}.$$

   (d) Make a smart choice for η to conclude the LDP upper bound for G.