Let $(X_n)_n$ be a discrete-time Markov chain on a finite space $\mathcal{X}$, with positive transition probability $(P_{xy})_{x,y\in\mathcal{X}\times\mathcal{X}}$, and define:

$$L^n := \tfrac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i}, \qquad \text{(empirical/occupation measure)}$$

$$L^{2,n} := \tfrac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(X_i,X_{i+1})}, \qquad \text{(pair empirical measure)}$$

$$L^{2,n,\text{per}} := \tfrac{1}{n}\sum_{i=1}^{n-1} \mathbb{1}_{(X_i,X_{i+1})} + \tfrac{1}{n}\mathbb{1}_{(X_n,X_1)}. \qquad \text{(periodic pair empirical measure)}$$

1. Use Stirling's formula to formally calculate, for $\nu^n \to \nu$,

$$\lim_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}(L^{2,n,\text{per}} = \nu^n).$$

2. Use the fact that the pair empirical measure $L^{2,n}$ satisfies a large-deviation principle to show that the occupation measure L^n satisfies a large-deviation principle in $\mathcal{M}_1(\mathcal{X})$, with some rate functional J. Is J good?

3. In fact, both results are also true when the transition matrix is not necessarily positive, only the rate functional must be slightly changed. Use the fact that occupation measures L^n on a finite space satisfy a large-deviation principle with some good rate functional J to show that the pair empirical measure $L^{2,n}$ satisfies a large-deviation principle.