1. Let $(\mu_n)_n$ satisfy a large-deviation principle in a metric space $\mathcal{X}$ with good rate functional $I : \mathcal{X} \to [0, \infty]$. Given $f \in C_b(\mathcal{X})$, define the tilted measure, for any measurable set $A \subset \mathcal{X}$:

$$\nu_n(A) := \frac{1}{Z_n} \int_A e^{nf(x)} \, \mu_n(dx), \qquad Z_n := \int_{\mathcal{X}} e^{nf(x)} \, \mu_n(dx).$$

Show that $(\nu_n)_n$ satisfies a large-deviation principle. What is the rate functional J?

- *Hint 1: the proof of the Varadhan Lemma also applies when the full space $\mathcal{X}$ is replaced by an arbitrary measurable set A.*
- *Hint 2: is the tail condition really needed?*
- *Validity check: is* $\inf J = 0$*?*

Lemma 0.1 (Laplace principle on $\mathbb{R}$)**.** *For any measurable set $A \subset \mathbb{R}$ and measurable function $g : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}} e^{-g(x)} \, dx < \infty$,*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \int_A e^{-ng(x)} \, dx = - \operatorname*{essinf}_{x \in A} g(x).$$

Definition 0.2. *A sequence $(\mu_n)_n$ satisfies a large-deviation principle in a topological space $\mathcal{X}$* with speed γ_n *and rate functional $I : \mathcal{X} \to [0, \infty]$ whenever:*

- $\lim_{n\to\infty} \gamma_n = \infty$,
- $\liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(U) \geq - \inf_U I$ *for all open $U \subset \mathcal{X}$,*
- $\liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(G) \leq - \inf_G I$ *for all closed $G \subset \mathcal{X}$,*
- *I is lower semicontinuous.*

2. Let $\mu_n(dx) := \sqrt{\frac{\log n}{\pi}} n^{-x^2} \, dx$.

   (a) Prove the large-deviation principle for μ_n, for a speed that gives a non-trivial rate functional.
   Hint : the Laplace principle in $\mathbb{R}$ really does not need finite Lebesgue measure, see e.g. Wikipedia on Laplace principle.

   (b) What is the rate functional for the large-deviation principle with speed n?