1. Let W_t be a d-dimensional Brownian motion [1] with $W_0 = 0$ a.s.. Let $X^n_t := \frac{1}{\sqrt{n}} W_t$.

   (a) Write down, for arbitrary times $0 < t_1 < t_2$, $x \in \mathbb{R}^d$, measurable set $A_2 \subset \mathbb{R}^d$, and fixed $n \in \mathbb{N}$,
   $$\nu^n_{t_1,t_2}(A_2|x) := \mathbb{P}(X^n_{t_2} \in A_2 \mid X^n_{t_1} = x).$$

   (b) What is the large-deviation rate functional of $\nu^n_{t_1,t_2}(\cdot|x)$ as $n \to \infty$ (with speed n) and fixed $0 < t_1 < t_2, x \in \mathbb{R}^d$?

   (c) Use the Markov property to write down, for $0 < t_1 < \dots t_M$, $M \in \mathbb{N}$, and measurable sets $A_1, \dots A_M \subset \mathbb{R}^d$, and fixed $n \in \mathbb{N}$:
   $$\mu^n_{t_1,\dots,t_M}(A_1 \times \dots \times A_2) = \mathbb{P}(X^n_{t_1} \in A_1, \dots, X^n_{t_M} \in A_M).$$
   (It could simplify notation to use integration variables $x_1, \dots, x_M$ and to set $x_0 := 0$ and $t_0 := 0$).

   (d) Use the Laplace principle to prove the large-deviation principle of $\mu^n_{t_1,\dots,t_M}$.
   (In this finite-dimensional setting it is easy to see that the rate function is indeed good...)

   (e) Use the Dawson-Gärtner Theorem to prove the large-deviation principle of the path measure
   $$\mu^n(B) := \mathbb{P}\big((X^n_t)_{t\in[0,\infty)} \in B\big),$$
   for $B \in \mathbb{R}^{d^{[0,\infty)}}$, equipped with the product topology.

   (f) Validity check: what is the deterministic limit of X^n, and is the rate functional 0 in this limit?

   In fact, it can be shown that the supremum over times is a limit where $M \to \infty$ and $\sup_{i \le M} |t_i - t_{i-1}| \to 0$.

   (g) For a smooth path $x : [0,\infty) \to \mathbb{R}^d$, use $\frac{x_{t_i} - x_{t_{i-1}}}{t_i - t_{i-1}} \approx \dot{x}(t_i)$ and the Riemann integral to formally simplify the rate functional.

   Compare this rate functional to "Schilder's Theorem", see for example König, Satz 2.3.1, or Dembo & Zeitouni, Th. 5.2.3.

[1] Probabilistic literature usually takes Brownian motion with generator $f \mapsto \frac{1}{2}\Delta f$, whereas in analytic literature $f \mapsto \Delta f$ is more common. So if you are a factor 2 off in this exercise, it's probably due to that.