Introduction to vine copulas

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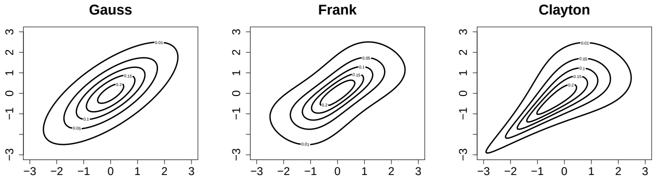
Motivation

- Copulas model marginal and common dependencies separately.
- There is a wide range of parametric copula families:



- But: Standard multivariate copulas
  - can become inflexible in high dimensions.
  - do not allow for different dependency structures between pairs of variables.

$\Rightarrow$ **Vine copulas** for higher-dimensional data.

Overview Vines

Vine pair-copulas

- **Bivariate copulas** are building blocks for higher-dimensional distributions.
- The dependency structure is determined by the bivariate copulas and a **nested set of trees**.

→ Vine approach is more flexible, as we can select bivariate copulas from a wide range of (parametric) families.

Model estimation

1. graph theory to determine the dependency structure of the data
2. statistical inference (maximum-likelihood, Bayesian approach ...) to fit bivariate copulas.

Background - Bivariate Copulas

Bivariate Copula

A bivariate copula function

$$C \quad : \quad [0,1]^2 \to \mathbb{R}$$

is a distribution on $[0,1]^2$ with uniform marginals.

Let F be a bivariate distribution with marginal distributions F_1, F_2.

Sklar's Theorem (1959)

There exists a two dimensional copula $C(\cdot,\cdot)$, such that

$$\forall (x_1, x_2) \in \mathbb{R}^2 : \quad F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \, .$$

If F_1 and F_2 are continuous, the copula C is unique.

Copula densities

Copula density (2-dimensional)

$$c_{12}(u_1, u_2) = \frac{\partial^2 C_{12}(u_1, u_2)}{\partial u_1 \partial u_2}$$

This implies

- joint density

$$f(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2)$$

- conditional density

$$f(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)$$

Important: pair-copula constructions

We can represent a density $f(x_1, \ldots, x_d)$ as a product of pair copula densities and marginal densities!

Example: $d = 3$ dimensions. One possible decomposition of $f(x_1, x_2, x_3)$ is:

$$f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2) f_{2|1}(x_2|x_1) f_1(x_1)$$

$$f_{2|1}(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)) f_2(x_2)$$

$$f_{3|12}(x_3|x_1, x_2) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) f_{3|2}(x_3|x_2)$$

$$f_{3|2}(x_3|x_2) = c_{23}(F_2(x_2), F_3(x_3)) f_3(x_3)$$

$$\begin{aligned} f(x_1, x_2, x_3) &= f_3(x_3) f_2(x_2) f_1(x_1) \text{ (marginals)} \\ &\times c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \text{ (unconditional pairs)} \\ &\times c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \text{ (conditional pair)} \end{aligned}$$

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Pair-copula construction (PCC) in d dimensions

Joe (1996), Bedford and Cooke (2001),Aas et al. (2009), Czado (2010)

$$f(x_1,\ldots,x_d) = \underbrace{\prod_{j=1}^{d-1}\prod_{i=1}^{d-j} c_{i,(i+j)|(i+1),\cdots,(i+j-1)}}_{\text{pair copula densities}} \cdot \underbrace{\prod_{k=1}^{d} f_k(x_k)}_{\text{marginal densities}}$$

with

$$c_{i,j|i_1,\cdots,i_k} := c_{i,j|i_1,\cdots,i_k}(F(x_i|x_{i_1},\cdots,x_{i_k}),(F(x_j|x_{i_1},\cdots,x_{i_k}))$$

for $i, j, i_1, \cdots, i_k$ with $i < j$ and $i_1 < \cdots < i_k$.

Remarks:

- The decomposition is <u>not</u> unique.
- Bedford and Cooke (2001) introduced a graphical structure called regular vine structure to help organize them.

Important: regular vine structure

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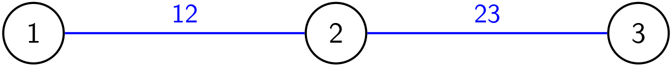

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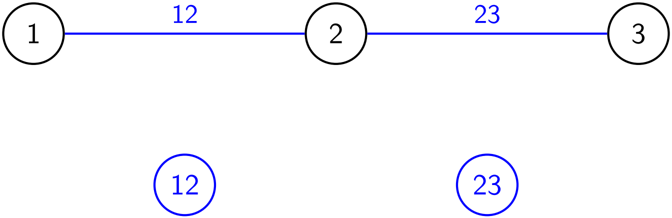

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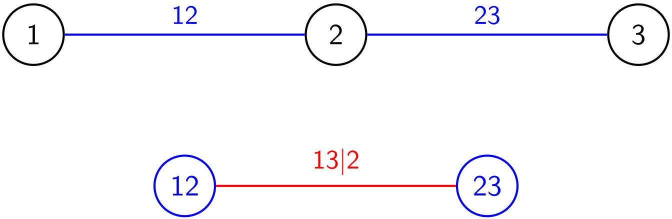
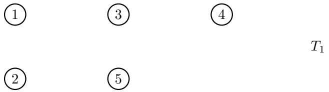


R-vine structure ($d = 5$) ▸ formal definition



Pair-copulas:

1. $C_{12}, C_{13}, C_{34}, C_{34}, C_{15}$
2. proximity condition If two nodes are joined by an edge in tree $j+1$, the corresponding edges in tree j share a node.
3. $C_{23|1}, C_{14|3}, C_{35|1}$
4. $C_{24|13}, C_{45|13}$
5. $C_{25|134}$

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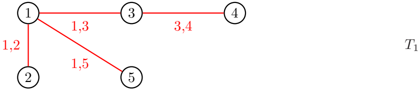

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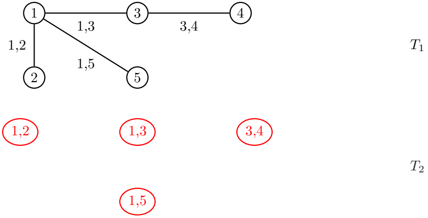

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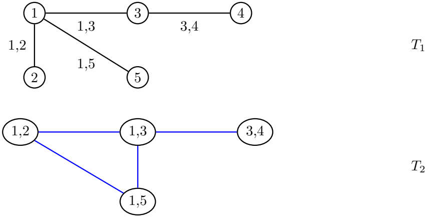

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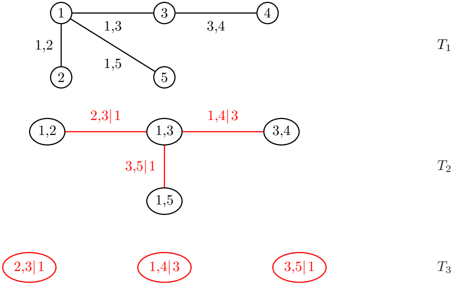

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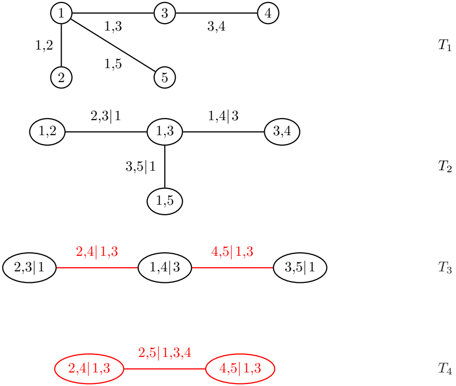

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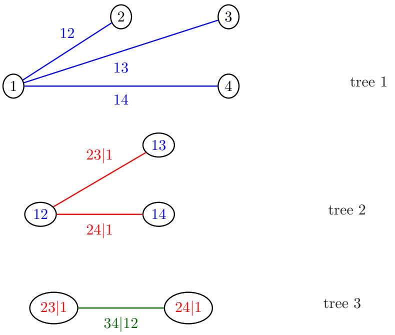
C-anonical vines

Each tree has a unique node that is connected to all other nodes.

$$f_{1234} = \underbrace{f_1 \cdot f_2 \cdot f_3 \cdot f_4}_{\text{nodes in } T_1} \cdot \underbrace{c_{12} \cdot c_{13} \cdot c_{14}}_{\substack{\text{edges in } T_1 \\ \text{nodes in } T_2}} \cdot \underbrace{c_{23|1} \cdot c_{24|1}}_{\substack{\text{edges in } T_2 \\ \text{nodes in } T_3}} \cdot \underbrace{c_{34|12}}_{\text{edge in } T_3}$$



D-rawable vines

Each tree is a path.

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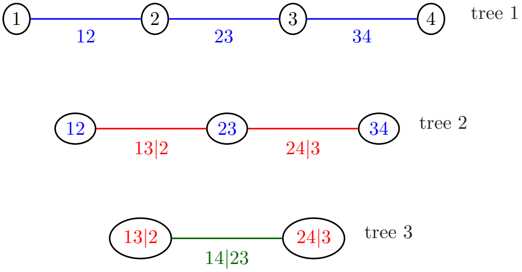

Preliminary summary: pair-copula decomposition

So far

Given a d-dimensional density, we can

- decompose it into products of marginal densities and bivariate copula densities.
- represent this decomposition with nested set of trees that fulfill a proximity condition.

Question

Given data, how can we estimate a pair-copula decomposition?

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Model = structure (trees) + copula families + copula parameters

Use our software package **CDVine**!
(Brechmann and Schepsmeier (2011))

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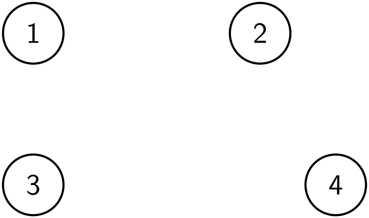

Data

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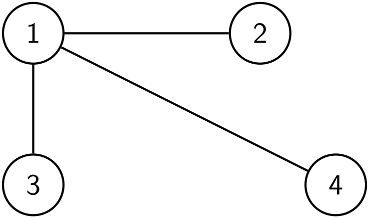

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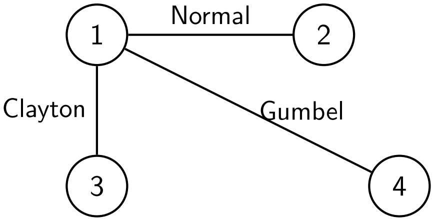

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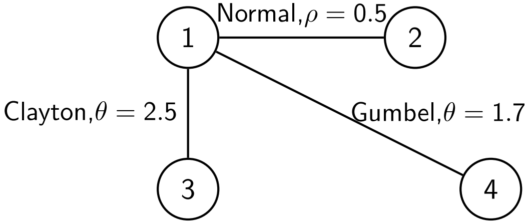

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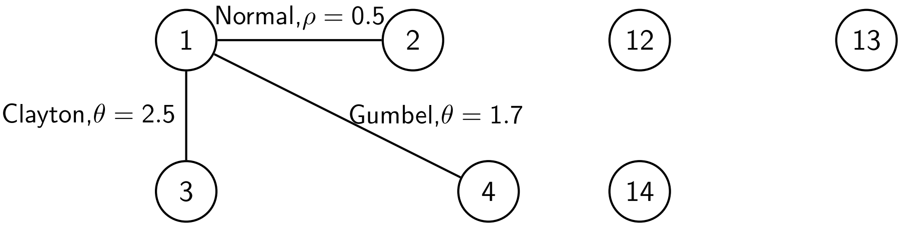

Pseudo observations

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Model selection and parameter estimation

Model = structure (trees) + copula families + copula parameters

Problem:

- Huge number of possible vines → structure selection
- $\frac{d(d-1)}{2}$ pair-copulas → copula selection

→ parameter estimation

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Structure selection

Possible edge weights

- Kendall's τ
- Spearman's ρ
- p-values of Goodness-of-Fit tests
- distances

Model selection

is done tree by tree via

- optimal C-vines structure selection (Czado et al. (2011))
- Traveling Salesman Problem for D-vines
- Maximum Spanning Tree for R-vines (Dissmann et al. (2011))
- Bayesian approaches (Reversible Jump MCMC)

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Copula selection

Copula selection

can be done via

- Goodness-of-fit tests
- Independence test
- AIC/BIC
- graphical tools like contour plots, λ-function, ...

Possible copula families

- Elliptical copulas (Gauss, t-)
- one-parametric Archimedean copulas (Clayton, Gumbel, Frank, Joe,...)
- two-parametric Archimedean copulas (BB1, BB7,...)
- rotated versions of the Archimedean for neg. dependencies
- ...

Parameter estimation

Estimation approaches:

- **Maximum likelihood estimation**
- **Sequential estimation:**
  - Parameters are estimated sequentially starting from the top tree.
  - Parameter estimates can be used to define pseudo observations for the next tree
  - Parameter estimation via $\theta = f(\tau)$ or bivariate MLE
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Applications

Dimensionality of applications

- Gaussian vines in arbitrary dimensions (Kurowicka and Cooke 2006)
- First non Gaussian D-vine models using joint maximum likelihood in 4 dimensions
- Bayesian D-vines with credible intervals in 7 and 12 dimensions
- Joint maximum likelihood now feasible in 50 dimensions for R-vines
- Sequential estimation of R-vines in 100 dimensions
- Sequential estimation for $d \gg 100$ dimensions with truncation (i.e. higher order trees only contain independent copulas)
- Heinen and Valdesogo (2009) sequentially fit a C-vine autoregressive model in 100 dimensions

Application areas:

- finance
- insurance
- genetics
- health
- images
- ...

Extensions (Projects of our research group)

Special vine models:

- vine copulas with time varying parameters
- regime switching vine models
- non parametric vine pair copulas
- Non Gaussian directed acyclic graphical (DAG) models based on PCC's
- discrete vine copulas
- truncated and simplified R-vines
- spatial vines
- copula discriminant analysis

Summary and outlook

- PCC's such as C-, D- and R-vines allow for very flexible class of multivariate distributions
- Efficient parameter estimation methods are available for dimensions up to 50
- Model selection of vine tree structures and pair copula types for regular vines still needs further work
- Efficient distance measures between vine distributions would be useful

Aas, K., C. Czado, A. Frigessi, and H. Bakken (2009).
Pair-copula constructions of multiple dependence.
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Reading material, software and current projects:

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Regular vine distribution

An d-dimensional regular vine is a sequence of d-1 trees

1. tree 1
   - d nodes: $X_1, \dots, X_d$
   - $d-1$ edges: pair-copula densities between nodes $X_1, \dots, X_d$
2. tree j
   - $d+1-j$ nodes: edges of tree $j-1$
   - $d-j$ edges: conditional pair-copula densities

- **Proximity condition:** If two nodes in tree $j+1$ are joined by an edge, the corresponding edges in tree j share a node.

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